6. The problem of finding the characteristics of the frequency function of y/x, in case the characteristics of the correlation or frequency function of x and y are known, has been studied only by Karl Pearson. He gave up his results as "practically unworkable" in case x and y are correlated. Under certain "almost necessary" restrictions the author developed formulas which are workable using the method of semi-invariants. The application of the results was illustrated numerically in two cases in which the correlation surface $z=F(x, y)$ is normal.

7. Professor Hopkins gave a report of an investigation now being conducted under his direction on periodic orbits for asteroids which revolve about the sun an integral number of times while Jupiter completes one revolution. The method of attack is similar to that used by Sir George Darwin in England and Professor F. R. Moulton in this country in problems of analogous nature. The results so far secured in this investigation were announced, but Professor Hopkins plans to develop the problem much further and to publish the results upon completion.

W. W. DENTON, *Secretary*

THIRTEENTH ANNUAL MEETING OF THE OHIO SECTION

The thirteenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 5, 1928, in connection with the meetings of the Ohio College Association. Chairman C. H. Yeaton presided over the afternoon session. Following the dinner in the evening there was an informal discussion of the question, "What shall we do for our students of superior ability?" Professor E. H.Clarke, chairman of the program committee, presided over this discussion.

Forty-six persons registered attendance, among whom were the following thirty-one members of the Association; J. P. Albert, R. B. Allen, W. E. Anderson, L. C. Bagby, L. Battig, Grace M. Bareis, P. E. Baur, H. M. Beatty, J. B. Brandeberry, C. T. Bumer, W. D. Cairns, V. B. Caris, F. E. Carr, E. H. Clarke, R. Crane, O. L. Dustheimer, E. E. Erickson, H. Hancock, Marie M. Johnson, H. W. Kuhn, C. C. Morris, R. L. Newlin, J. Pierce, S. E. Rasor, C. E. Rhodes, Hortense Rickard, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, B. F. Yanney, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, E. H. CLARKE; Secretary-Treasurer, RUFUS CRANE; Member of Executive Committee, S. A. ROWLAND; Member of Program Committee, LOUIS BRAND. It is expected that the next meeting will be held at the Ohio State University on Thursday, April 4, 1929.

Three papers were presented, titles and abstracts of which follow:

1. "The construction of the tangent to certain curves" by the Chairman, Professor C. H. YEATON, Oberlin College.

2. "The analogy of the theory of Kummer's ideal numbers with chemistry and its prototype in Plato's concept of idea and number" by Professor HARRIS HANCOCK, University of Cincinnati.

3. "Some methods to be used in obtaining rapidly converging power series for certain root approximations" by Professor E. H. CLARKE, Hiram College.

1. To the early mathematician the determination of the tangent at an arbitrary point of a curve was a geometrical problem; general principles and methods of attack were lacking. With the invention of the geometry of Descartes, the synthetic method gave way to the analytic; early definitions were modified and the solution gained in generality. In the present paper Professor Yeaton reviews a geometrical construction of the tangent which has been developed by Jules Poivert, Professor of Architecture, École Polytechnique, Montreal. In many cases the construction is purely geometric; in others, the position of the tangent to some fundamental curve is assumed.

2. Mathematics, occupying in its realm an intermediate position between the realm of the natural sciences and the realm of abstract reason, is both a means of computation and a medium of thought. The mathematical realm so overlaps the two adjacent realms that a transition from the one to the other through it may give rise to confusion both in term and in concept. With Plato there appear two kinds of number, the mathematical numbers consisting of homogeneous unities which can therefore be reckoned together, each with each, and the ideal numbers not identical with ideas but intermediate between them and the things of sense. Accordingly the mathematical numbers express quantitative, the ideal numbers logical determinations. This appears to be the interpretation of Zeller. Prof. Paul Shorey seems in a measure at variance with the Zeller interpretation.

Professor Hancock holds that if the Plato ideal number is rightly interpreted, there is no incorrectness in either the Zeller or the Shorey contention; and he asserts that the theory of algebraic ideal numbers of Kummer is the key for the solution of the difficulty. Its analogy with chemistry offers convincing evidence.

Professor Hancock presents his thesis in a form that can be read by philosophers as well as by mathematicians. Two illustrations of Sommer and Hensel are used to introduce the notion of the ideal number. This notion is generalized for the mathematicians by means of an example taken from an algebraic realm. The Dedekind ideals and modals are introduced and the correlation with the quadratic forms is pointed out. The analogy with chemistry is drawn from the

paper of Kummer in Crelle's Journal, vol. 35, p. 359 (1846) and references are given to the works of Shorey and Zeller.

3. Professor Clarke showed how the binomial series could be made very rapidly converging for the computation of the roots of numbers by the introduction of close rational approximations to the roots, obtained from continued fractions.

RUFUS CRANE, *Secretary*

THE FIFTH ANNUAL MEETING OF THE NEBRASKA SECTION

The fifth annual meeting of the Nebraska section of the Mathematical Association of America was held on April 28, 1928, at Midland College, Fremont, Nebraska. Twenty persons were present including six members of the Association. The following program was presented:

1. "Reducents, or values of the variable for which a polynomial with integer coefficients can not represent a prime" by Mr. O. C. COLLINS, University of Nebraska.

2. "A standard curve for grades in college mathematics" by Professor H. C. FEEMSTER, York College. (By title.)

3. "The borrower's rate in building and loan associations" by Professor C. C. CAMP, University of Nebraska.

4. "A finite projective geometry" by Miss GLADYS BALDWIN, University of Nebraska.

Abstracts of these papers follow:

1. Mr. Collins showed that when certain functions of the coefficients are substituted for the variable in a polynomial with integer coefficients the result is the product of factors which are functions of the coefficients. For integer values of the variable corresponding to such substitutions the polynomial will in general represent a composite number.

3. Professor Camp reviewed various methods for computing rates of interest, among them a new one which is both simple and accurate. A random sample was taken from associations in Illinois, and the rate of the borrower correlated with that of the investor. The average rates were 8.87 percent and 6.87 percent respectively, contrary to popular opinion.

4. Miss Baldwin made a study of the finite projective geometry in which each line consists of eight points. She enumerated the number of involutions on the line, the number of conics in the plane and developed a number of theorems characteristic of this particular geometry.

Officers elected for the year 1928–29 are: H. C. FEEMSTER, York College, Chairman; ELLEN H. FRANKISH, North High School, Omaha, Secretary-

Treasurer; J. M. HOWIE, Nebraska Wesleyan University, Member of Executive Committee.

ELLEN H. FRANKISH, *Secretary*

THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-third regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at the United States Naval Academy, Annapolis, Maryland, on Saturday, May 5, 1928. Sessions were held in the morning and afternoon, Professor J. R Musselman, Chairman of the Section, presiding at both sessions.

About sixty persons attended the meeting, including the following forty-seven members of the Association: O. S. Adams, R. N. Ashmun, Clara L. Bacon, G. A. Bingley, L. M. Blumenthal, C. C. Bramble, J. A. Bullard, Paul Capron, G. R. Clements, A. B. Coble, A. Cohen, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, J. B. Eppes, P. J. Federico, Michael Goldberg, Harry Gwinner, W. M. Hamilton, P. E. Hemke, L. S. Hulburt, F. E. Johnston, H. P. Kaufman, L. M. Kells, W. D. Lambert, A. E. Landry, Bessie I. Miller, A. K. Mitchell, T. W. Moore, Frank Morley, F. D. Murnaghan, J. R. Musselman, J. W. Peters, E. C. Phillips, C. H. Rawlins, Jr., J. N. Rice, A. W. Richeson, H. M. Robert, Jr., R. E. Root, J. B. Scarborough, W. F. Shenton, Marian M. Torrey, John Tyler, W. J. Wallis, Paul Wernicke, Elizabeth W. Wilson, E. W. Woolard.

In persuance of their longstanding tradition for generous hospitality the Annapolis members entertained those attending the meeting at luncheon during the interval between the two sessions. A hearty vote of thanks was passed in appreciation of this hospitality and of the courtesy of the authorities of the Academy for their welcome to the Section within its walls.

At the beginning of the afternoon session a business meeting was held at which the committee on nominations, consisting of Dr. Adams, Professor Cohen, and Professor Root, presented the following list of nominees: For chairman, Professor C. C. BRAMBLE; for secretary, Rev. E. C. PHILLIPS; as members of the executive committee, Professor HARRY GWINNER and Dr. L. M. BLUMENTHAL. No other nominations were offered and the election of the above named candidates was made unanimous.

The following eight papers were presented:

1. "A transformation of the Clifford configuration into itself" by Professor FRANK MORLEY, Johns Hopkins University.

2. "A cubic curve and a reflector" by Dr. BESSIE I. MILLER.

3. "An interesting conformal projection of the sphere depending upon elliptic functions" by Dr. OSCAR S. ADAMS, U. S. Coast and Geodetic Survey.

4. "Two mutual relations possible between a pair of tetrahedra" by Dr. PAUL WERNICKE, U. S. Patent Office.

5. "The first English Euclid" by Professor WALTER F. SHENTON, American University.

6. "Some researches on the triangularly symmetric rational quartic" by Mr. FRANK E. SMITH, Catholic University of America (introduced by Professor A. E. Landry.)

7. "On the genetic equilibrium of a random bred population" by Professo· TOBIAS DANTZIG, University of Maryland, and Professor WILLIAM B. KEMP, University of Maryland, (introduced by Professor Dantzig).

8. "An invariant of a space configuration" by Dr. LEONARD M. BLUMENTHAL, Johns Hopkins University.

The papers led to rather extensive discussion in which the following members took part: Messrs. Blumenthal, Capron, Coble, Dantzig, Gwinner, Lambert, Morley, Murnaghan, Wernicke, and Miss Wilson.

Abstracts of these papers follow:

1. A Clifford configuration is the figure of n lines of a plane, say $1, 2, \cdots, n$, completed. That is the points 12, 13, 23 are on a circle 123. The circles 123, 234, 341, 412 are on a point 1234. The points such as 1234, 2345, 3451, $\cdots$, are on a circle 12345. And so on. There are then, including ∞, which is here a point, 2^{n-1} points and 2^{n-1} circles; there are $n-1$ points on each circle and $n-1$ circles on each point. Professor Morley indicated transformations T_n of a C_n into itself in the simpler cases. For C_3, the transformation T_3 is focal pairing or transformation by isogonal conjugates. The transformation T_4 is quadratic involution or polarity, $xy=k$. The transformation T_5 is a contact transformation reducible to $xy=1+\mu t$, where $|t|=1$. The transformation T_6 is a Cremona transformation with four singular points (in the inversive plane). Only the points are sent into themselves in this case. In canonical form T_6 is $\sum^4[\lambda_i/(x-a_i)]=0$, where $\sum\lambda_i=0$ and the values of λ_i are real.

2. A physician, Dr. Wood, of Cleveland, the inventor of one of the forms of reflectors much used in automobile lamps, has made a modification of that reflector. A plane section of the new reflector is a portion of a nodal cubic. Commercially a parabola which approximates this portion of the cubic will be used as the generating curve of the surface of the reflector. This work is covered by a patent.

3. This paper described the method of computation employed in placing the whole sphere within a square with the poles situated on one of the diagonals

and symmetrically placed with respect to the center of the square. The calculations were made for the ten degree intersections and a copy of the constructed projection was shown. The method of attack employed was aimed to simplify the computations as much as possible so that they can be made by one who has a slight knowledge of trigonometry with the use of tables. No further use was made of elliptic functions than simple interpolations in Legendre's table of the first elliptic integral for $k=1/\sqrt{2}$.

4. A quadrilateral in space S_3, or a "tetragram" T, defines a tetrahedron the faces of which, in (reversible) cyclic order, are the planes laid through pairs of consecutive sides of T. Four vectors representing these faces, i.e. perpendicular to them and proportional to their areas, form when placed end to end, a second tetragram T'. Starting with T' we find T, or a tetragram similar to T, by the same process. T and T' are here called *orthogrammatic* to one another. The same term when applied to the tetrahedra, defines a 3 to 1 reciprocal relation.

Steiner has applied the term *orthologiques* to two triangles t, t' if the perpendiculars from the vertices of t to the sides of t' are concurrent. He proves that the perpendiculars from the vertices of t' to the sides of t then also concur. A proof by vector algebra is here given and proves applicable to tetrahedra T, T' or to *simplicia* [$(n+1)$-points] in any S_n. If the perpendiculars from the vertices of T to the faces of T', and vice versa, concur, T and T' may therefore be called *orthologic*.

5. A description, with illustrations, of the two folio volumes of "The Elements of Geometrie of the most auncient Philosopher Euclide of Megara, Faithfully (now first) translated into the Englishe toung, by H. Billingsley, Citizen of London." This interesting work is from the press of John Daye and was printed in London in 1570. A compendious preface by John Dee adds to the interest of the work, as do the figures in Book Eleven, which are made of printed papers, so pasted and folded into the work that they may be unfolded to make actual models for the space theorems. These volumes, together with many others of intense historical interest, are a part of the mathematical library of the late Dr. Artemas Martin, mathematician and collector of Washington, D.C., which he donated to the American University of that city.

6. If the equations of the plane rational quartic are set up so that it has metric as well as projective symmetry, it has parametric value at the double-points: α, α^{-1}; $\omega\alpha$, $\omega\alpha^{-1}$; $\omega^2\alpha$, $\omega^2\alpha^{-1}$, where the parameter $\alpha=e^{i\theta}$ takes all values on the unit circle in the complex domain. The problem is: how many of the 8 possible triangles simultaneously in-and-circumscribed to this curve can be real. It is found that the 8 triangles break up into a set of two equilateral triangles and a set of six non-equilateral triangles. The sets appear either all together or not at all. The set of six is real for a very small range of the para-

meter when $\theta = 77° 45'$, approximately, and for the range $0° < \theta < 60°$. The set of two is real for the range $0° < \theta < 60°$ and for the additional range $2 < \rho < 3$ where $\rho = \alpha + \alpha^{-1}$. The "trioval" shape $(0 < \rho < 1)$ has, then, for a small range 6 real in-and-circumscribed triangles. The "loop" shape $(1 < \rho < 2)$ has always 8 real. The "acnodal" shape $(\rho > 2)$ has two real for the range $2 < \rho < 3$.

For no proper form of this curve does an equilateral triangle have a side in common with a non-equilateral triangle.

7. In this paper the Mendelian law is generalized by assigning general values (u, v, w) to the reproductive ratios of the dominant, heterozygote and recessive types of a species. The mathematical formulation of the law leads to the recurrence formula

$$(1) \qquad r_k \equiv (\alpha r_k + 1) r_k / (r_k + \beta), \text{ where } \alpha = u/v,\ \beta = w/v,\ r_k = a_k/b_k.$$

r_k is the dominance ratio for the kth generation. The question of equilibrium is then reduced to the study of iteration (1) and particularly of the limiting points of the set $r_1, r_2, r_3, \cdots$. It is found that for $\alpha < 1, \beta < 1$ the equilibrium is stable, the dominance ratio tending towards $r = (1-\alpha)/(1-\beta)$.

When $\alpha > 1, \beta > 1$, the equilibrium is unstable and the evolution results in the rapid extermination of either the dominant or of the recessive characteristic. Finally, the classical case $\alpha = \beta = 1$ is that of indifferent equilibrium.

The reproductive capacity is given by the quadratic form

$$P_{k+1} = u a_k^2 + 2v a_k b_k + w b_k^2$$

In the case of stable genetic evolution the function P reaches its maximum at equilibrium. Therefore a selective process based on the adjustment of the dominance ratio will invariably reduce the reproductive capacity of the species.

8. This paper is an extension to Euclidean spaces of higher dimensions of the results recently obtained for the plane. (L. M. Blumenthal, *Lagrange resolvents in Euclidean Geometry*, American Journal of Mathematics, vol. 49, 1927, pp. 511–522). It is shown that attached to an ordered $2n$-point in S_n there is a lineo-linear invariant under translations of either component n-point. This invariant (entirely analogous to the one developed in the paper above) is called the *norm* of the configuration and is expressed in terms of the Lagrange Resolvents of the vertices of the $2n$-point.

As a result of the vanishing of their norms, two interesting theorems are developed for the space 6-point and 8-point.

The norm for ordered points on a sphere is defined and a theorem for the ordered 6-point is deduced.

EDWARD C. PHILLIPS, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The regular spring meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Benedict, St. Joseph, Minnesota, on Saturday, May 19, 1928. At the request of the Chairman Sister Claudette, O.S.B., Professor H. H. Dalaker of the University of Minnesota presided.

The attendance was 65, including the following 23 members of the Association: Sister Alice Irene, W. O. Beal, R. W. Brink, W. H. Bussey, Elizabeth Carlson, H. H. Dalaker, J. M. Earl, Clara L. Hancock, W. L. Hart, H. E. Hartig, D. Jackson, W. H. Kirchner, W. D. MacMillan, M. Nordgaard, W. C. Risselman, R. R. Shumway, A. K. Solum, F. J. Taylor, Ella Thorp, A. L. Underhill, E. D. Wells, H. B. Wilcox, G. Winkelmann.

At the afternoon session the following officers were elected for the coming year: Chairman, Sister ALICE IRENE, College of St. Catherine, St. Paul; Secretary, A. L. UNDERHILL, University of Minnesota; an Executive Committee consisting of the Chairman, the Secretary, and H. E. HARTIG, University of Minnesota, MARIE NESS, University of Minnesota, and CLARA L. HANCOCK, Junior College, Virginia, Minnesota.

A motion was passed expressing the appreciation of the Section for the hospitality of the College of St. Benedict.

The following six papers were read:

1. "Some early applications of calculus to mechanics" by Professor W. D. MACMILLAN, University of Chicago.

2. "On the existence of certain approximating functions" by Professor W. C. Risselman, University of Minnesota.

3. "A course in mathematics for students of statistics" by Professor R. W. BRINK, University of Minnesota.

4. "Analytic and topological transformation of closed surfaces of genus p" by Dr. EDWARD SAIBEL, University of Minnesota.

5. "Prediction of student achievement in freshman mathematics" by Professor W. L. HART, University of Minnesota.

6. "Concerning integration by parts" by Professor H. E. HARTIG, University of Minnesota.

Abstracts of these papers follow:

1. Professor MacMillan gave an account of some of the early applications of the calculus to mechanics, and especially considered problems relating to the gravitational potential of spherical shells.

2. This note dealt with the problem of approximating to a given function

of a complex variable along a curve C by means of a linear combination of unitary bi-orthoganal functions so as to minimize the integral along C of the square of the absolute value of the error. The method of completing squares was used in proving existence and uniqueness. The whole purpose of the note was to call attention to this elementary method of proving an important well-known theorem.

3. In his paper Professor Brink described a course which he has been giving this year at the University of Minnesota. It is designed to give rapid preparation in the mathematics necessary for the study of the mathematical theory of statistics and for reading the literatures of statistics, to persons who have attained some maturity and cannot spare the time from work in their own fields to follow the ordinary courses in mathematics through the calculus.

4. It has been shown by Franklin and Wiener[1] that every continuous one-to-one transformation of a two dimensional region of finite connectivity may be approximated to an arbitrary degree of exactness by an analytic one-to-one transformation. It was the object of Dr. Saibel's paper to show that this theorem can be extended to closed surfaces.

5. The department of mathematics in the College of Science, Literature, and the Arts of the University of Minnesota has experimented with various devices for obtaining, early in each term, a prediction of the future achievement of students in freshman courses in mathematics. Professor Hart described the predictive devices used and discussed the application of the predictions in the placement work of the department of mathematics. This placement of students involves the elimination, at the end of two weeks, of the lowest one-sixth of the students in each course and the sectioning of the remainder on the basis of ability.

6. Professor Hartig showed by examples how one can often simplify integration by parts in Case I, in integrating the expression usually called dv, by throwing in the integration constant, which later is chosen for the purpose of simplification.

On the day preceding this meeting of the Minnesota Section, Professor MacMillan delivered a public lecture at the University of Minnesota on "The evolution and dissolution of matter."

A. L. UNDERHILL, *Secretary*

[1] Philip Franklin and Norbert Wiener, *Analytic approximations to topological transformations*, Transactions of the American Mathematical Society, vol. 28 (1926), pp. 762–785.

A SOURCE BOOK IN THE HISTORY OF MATHEMATICS

By DAVID EUGENE SMITH, Columbia University

Professor Gregory D. Walcott of Hamline University, St. Paul, Minnesota, has for some time been planning a series of source books in the history of the sciences, covering the period 1500–1900. To carry out this project the American Philosophical Society has recently secured from the Carnegie Foundation a grant sufficiently large to permit of a beginning in the publication of the series. It is proposed to use this sum to underwrite the publication of the first one or two volumes, and to use the income from the sales to allow for the publication of others. It is expected that the second volume will relate to mathematics. It will be edited by a committee consisting of Professors R. C. Archibald, Florian Cajori, and David Eugene Smith (chairman).

The general tentative plan for this volume is to publish articles of about 6–20 pages each, giving

(1) The title, author and source of the material selected.

(2) The name and position of the editor of the article.

(3) A brief statement, in a few lines, of the reason why the material has been considered important enough to have place in the publication.

(4) The original text, transcribed and checked with great care.

(5) A translation into English, in case the original is in a foreign language.

(6) A brief explanation, by notes or commentary, of any obscure statements needing elucidation. This feature must necessarily be very limited, the space at the Committee's disposal being devoted principally to the source material itself.

For example, if Cardan's solutions of the cubic and biquadratic equations were given, the Latin text would probably fill about four pages, the translation would take four more, and the explanatory notes with brief references to the work of Del Ferro, Tartaglia, and Ferrari might take one more. In any case, if the articles averaged twelve pages, only about forty could be used.

There are no funds with which to pay for the contributions. As with the members of the committee, a contributor must make the work a "labor of love," the publication of his name with the article which he contributes being the only other reward. There is barely enough money available to assume a beginning of the publication, all returns from sales being devoted to extending the series. No doubt any balance at the close of the project, some years hence, would be equally divided, but in any case it could hardly be worth considering.

It is desirable that all members of the American Mathematical Society and of the Mathematical Association of America should feel the importance of the undertaking and that those who have suggestions as to source material of value to college instructors, to graduate students, and to those high school

teachers who may desire to study the origins of the science, should send them to the committee at an early date. What is much more important, however, is that members should, if possible, cooperate in the work by volunteering to transcribe, translate, and annotate certain classical memoirs or extracts from treatises, these being confined to material of undoubted value as foundations upon which various parts of the mathematical structure stand.

Nearly two hundred topics have already been suggested by a few scholars to whom the matter has been informally mentioned. The list is too long to be published in full, and it is evident that only a small fraction of the items can be finally selected. The Committee would be glad, however, to receive as many further suggestions as possible.

It is desirable that

(1) Any piece of source material chosen should be of general interest to mathematicians, or at any rate to a large enough group to warrant the choice.

(2) It should not be already conveniently accessible in books that can be found in university or large public libraries.

(3) If possible, it should be of a nature to permit of printing as a whole, and should therefore be in concise form. This does not exclude the publication of definite chapters of a book or memoir, but a mere series of brief extracts would be better adapted to a series of historical essays than to a source book.

With these points in view the Committee asks for suggestions as to

(1) A limited number of the most important and epoch-making memoirs or definite sections of treatises, of interest to a sufficiently large number of readers, written during the period 1500–1900, and not generally accessible.

(2) A statement of the precise location of such source material; of the probable number of octavo pages (say of 360 words each) required for (a) the original text and (b) the translation; and of the probable necessity for explanatory notes.

(3) Whether or not the correspondent will volunteer to supply the text, translation, and notes.

The Committee volunteers its services to collate all replies and to follow as far as possible the desires of the majority of those who express an interest in the project. It must, of course, reserve the right to select among several volunteers who choose the same subject, and to make final choice as to the publication of the articles submitted. With such a large range of topics and a possible large number of expressed opinions and offers to collaborate, this selection will naturally take some time, but it is hoped to expedite the work as rapidly as circumstances permit It is probable that articles for which there is no space in the volume will be welcomed by various periodicals.

In case this volume meets with general approval, it is the expectation that two other volumes on mathematics may be published later; one on source

material before 1500, and the other on such material produced since 1900 probably with other important memoirs omitted from the present volume through lack of space.

No selections have as yet been made, either of topics or contributors, and it is hoped that volunteers will notify the Committee as soon as possible of any article or articles that they will be able to prepare.

It is desirable to bring this matter to the attention of mathematical clubs of various kinds in the graduate schools of our universities, thus enlisting the interest of the advanced students in mathematical courses. It is probable that a considerable number of the articles will be written by men and women of this type.

Suggestions of topics and, what is even more important, offers to edit certain ones in which the volunteer is especially interested, may be sent to Professors R. C. Archibald, Brown University, Providence, R. I.; Florian Cajori, Berkeley, Calif.; or David Eugene Smith, 501 West 120th Street, New York City.

A few possible topics are the following: decimal fractions; Delamain and Oughtred on the slide rule; the binomial theorem; mathematical induction; Cardan's solution of the cubic and biquadratic (Del Ferro, Tartaglia, Ferrari); De Moivre's theorem $\cos nx+i \sin nx=(\cos x+i \sin x)^n$ and the relation of Cotes, $ix=\log(\cos x+i \sin x)$; definitions of e; uniform convergence; Determinants; Hill-Poincaré infinite determinants; Kummer's ideals; Descartes, selections from *La Géométrie;* Desargues's theorem on perspective triangles; Gauss's statement on the construction of regular polygons; Bolyai's first essay (1823); Lobachevsky's *Pan-Geometry* (1835); circular points at infinity; Gauss's measure of curvature of surfaces; Bolzano (Weierstrass) nowhere differentiable continuous functions; Bessel's functions and numbers.

THE ANALOGY OF THE THEORY OF KUMMER'S IDEAL NUMBERS WITH CHEMISTRY AND ITS PROTOTYPE IN PLATO'S CONCEPT OF IDEA AND NUMBER[1]

By HARRIS HANCOCK, University of Cincinnati

As this article is intended for those who are interested in philosophic as well as mathematical problems, it may be well to indicate the nature of Kummer's ideal numbers by means of two examples. The first is due to Sommer, the second to Hensel. The presentation of the ideal numbers which is given immediately after these examples is not necessary for a casual survey of the subject under consideration. The mathematician, however, who reads the entire article, will, I believe, get a deeper insight into the Plato concept.

[1] Read before the Ohio Section of the Mathematical Association of America, April 5, 1928.

The following is the example used by Sommer (*Vorlesungen über Zahlentheorie*). Consider as a fixed realm of rationality the realm composed of integers of the form $4n+1$ and permit in the discussion only the operations of multiplication and division in their usual sense. In the series

$$1,5,9,13,17,21,\cdots,\ 45,\cdots,\ 117,\cdots,\ 517,\cdots$$

it is clear that the product of any two integers of the series is an integer of the series since $(4m+1)\ (4n+1)$ is of the form $4g+1$, where m and n are any two positive integers and $g=4mn+m+n$. The numbers 5, 9, 13, 17, 21, 29, $\cdots$, are irreducible, that is, they take the place of prime integers in this realm, in that they are not equal to the product of any other two integers of the realm.

Next observe that the number 10857 may be factored in the following two different ways $10857=141\cdot 77=21\cdot 517$, where 21, 77, 141 and 517 are prime numbers in the fixed realm. Thus it is seen that in this realm the factorization of an integer into its prime factors is not a unique process. This factorization, as is well known, becomes unique, if the fixed realm in question is extended so as to comprise what is known as the realm of *all* rational integers, in which $10857=3\cdot 7\cdot 11\cdot 47$.

Kummer's thought when applied to the above special case, consists in replacing the factors 3, 7, 11, 47, by what may be called "ideal numbers" in the restricted realm. In this realm, observe that the integers 3, 7, 11, 47 as such do *not* exist. To grasp the import of the "ideal numbers," denote the greatest common divisor of two integers a and b by the symbol (a, b) and observe that $(a, b)=(b, a)$. In the extended realm (that is, the realm of all natural integers) 3 is the greatest common divisor of 21 and 141 and may be written $3=(21, 141)$, where 21 and 141 are entities in the restricted realm. Further, we may put

$$7=(21,77);\qquad 11=(517,77);\qquad 47=(517,141);$$

$$(141)=(141,21)(141,517);\quad 77=(77,21)(77,517);$$

and

$$(10857)=(141,21)(141,517)(77,21)(77,517)=141\cdot 77.$$

On the other hand,

$$(21)=(21,141)(21,77);\quad (517)=(517,77)(517,141);$$

and

$$(10857)=(21,141)(21,77)(517,77)(517,141)=21\cdot 517.$$

In both cases the factorization of 10857 through its "ideal" prime factors leads to a unique result.

Similarly, it is seen that $693=21\cdot 33=9\cdot 77$, where 21, 33, 9, 77 are irreducible in the restricted realm as is also $441=(21)^2=9\cdot 49$.

Another illustration due to Hensel[1] is of interest. Let all the natural integers be distributed into two classes. Into the class C_0 let unity and those integers enter which when factored offer an even number of prime factors, while the class C_1 is to include all those integers which when factored present an odd number of prime factors. It is seen that

$$C_0 = [1,4,6,9,10,14,15,16,21,22,24, \cdots]$$
$$C_1 = [2,3,5,7,8,11,12,13,17,18,19,20,23, \cdots].$$

Let only the integers of one of the classes, say C_0, form a fixed realm. We shall confine our attention only to integers of this realm. It is seen that $210 = 6 \cdot 35 = 10 \cdot 21 = 14 \cdot 15$ which are three products of prime integers in C_0. Denote the greatest common divisor of two integers a and b by the symbol (a, b) and observe that we may write

$$\begin{aligned} 210 &= (6,10)(6,21)(35,10)(35,21) = 2 \cdot 3 \cdot 5 \cdot 7, \\ &= (6,14)(6,15)(35,14)(35,15) = 2 \cdot 3 \cdot 7 \cdot 5, \\ &= (10,14)(10,15)(21,14)(21,15) = 2 \cdot 5 \cdot 7 \cdot 3. \end{aligned}$$

Observe further that the ideals in each of the last three lines are equal, for example, $(6, 10) = (6, 14) = (10, 14)$. If then we call the integers of C_0 the real integers and those of C_1 the ideal integers, it is seen that 210 is equal to the unique product of the four ideal (Kummer) integers 2, 3, 5, 7, which do not have a real existence in C_0. Notice also that the elements of the ideals, say 6, 10 of (6, 10) are numbers of the fixed realm C_0.

Consider next the factorization of 21 in the realm $R(\sqrt{-5})$, namely,

$$21 = 3 \cdot 7 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5}) = (4 + \sqrt{-5})(4 - \sqrt{-5}).$$

Integers of this realm are of the form $a+b\sqrt{-5}$, where a and b are rational integers. Clearly there is something *common* to 7 and to at least one of the factors $(1+2\sqrt{-5})$, $(1-2\sqrt{-5})$. Take the product of these factors and form the congruence

$$1 - 2^2(-5) \equiv 0 \pmod 7. \tag{1}$$

Note that $-5 \equiv 3^2 \pmod 7$, so that (1) becomes $1 - 2^2 \cdot 3^2 \equiv 0 \pmod 7$.

1°. From this it is seen that $1+2 \cdot 3 \equiv 0 \pmod 7$. Compare this congruence with the factor $1+2\sqrt{-5}$. Kummer denoted that which is common to 7 and $1+2\sqrt{-5}$ by what I call the *Kummer factor* $\{7, 3\} = K_1$, while the Kummer factor $K_2 = \{7, -3\}$ denotes what is common to 7 and $1-2\sqrt{-5}$.

---

[1] Hensel, *Gedächtnissrede auf E. E. Kummer*, Abhandlungen zur Geschichte der Mathematischen Wissenschaften, vol. 29 (1910), No. 22.

Observe in a similar manner the congruence

$$1 - 2^2(-5) \equiv 0 \pmod 3 \tag{2}$$

and note that $-5 \equiv 2^2 \pmod 3$. Writing the congruence (2) in the form $1-2^2 \cdot 2^2 \equiv 0 \pmod 3$, it is seen that $K_3 = \{3, 2\}$ and $K_4 = \{3, -2\}$ are the Kummer factors of 3 and $1+2\sqrt{-5}$ and of 3 and $1-2\sqrt{-5}$, respectively.

2°. Similarly, that which is common to 7 and $4+\sqrt{-5}$ is the Kummer factor $K_1 = \{7, 3\}$, while $K_2 = \{7, -3\}$ is the factor of 7 and $4-\sqrt{-5}$; further, $K_3 = \{3, 2\}$ is the Kummer factor of 3 and $4+\sqrt{-5}$, $K_4 = \{3, -2\}$ being that of 3 and $4-\sqrt{-5}$.

3°. To derive that which is common to each of the pairs $1+2\sqrt{-5}$, $4+\sqrt{-5}$; $1+2\sqrt{-5}$, $4-\sqrt{-5}$; $1-2\sqrt{-5}$, $4+\sqrt{-5}$; and to $1-2\sqrt{-5}$, $4-\sqrt{-5}$, we may proceed as follows:

Denote what is common to two numbers α and β by the symbol (α, β). Thus $(\alpha, \beta) = T$, means "what is common to α and β is T." From above

$$(7,1+2\sqrt{-5}) = K_1 = (7,4+\sqrt{-5}); \quad (7,1-2\sqrt{-5}) = K_2 = (7,4-\sqrt{-5});$$
$$(3,4-\sqrt{-5}) = K_3 = (3,1+2\sqrt{-5}); \quad (3,4+\sqrt{-5}) = K_4 = (3,1-2\sqrt{-5}).$$

Observe that

$$(4+\sqrt{-5},1+2\sqrt{-5}) = (4+\sqrt{-5},1+2\sqrt{-5},2[4+\sqrt{-5}]-[1+2\sqrt{-5}])$$

so that

$$(4+\sqrt{-5},1+2\sqrt{-5}) = (4+\sqrt{-5},1+2\sqrt{-5},7)$$
$$= (4+\sqrt{-5},7) = (1+2\sqrt{-5},7) = K_1.$$

Similarly,

$$(4-\sqrt{-5},1-2\sqrt{-5}) = (4-\sqrt{-5},7) = K_2 = (1-2\sqrt{-5},7).$$

Note that there is nothing save unity in common to $4+\sqrt{-5}$ and $4-\sqrt{-5}$. For, if there were, there would be something common to these two quantities and their sum, which is 8. As there is something in common with either of these quantities, say $4+\sqrt{-5}$ and 21, it would follow that there is something in common with

$$(21,8,4+\sqrt{-5}) = (21,8,8\cdot 8 - 3\cdot 21, 4+\sqrt{-5}) = (1,4+\sqrt{-5}) = (1),$$

contrary to the hypothesis.

Next observe that

$$(4+\sqrt{-5},1-2\sqrt{-5}) = (4+\sqrt{-5},1-2\sqrt{-5},2[4+\sqrt{-5}]$$
$$+[1-2\sqrt{-5}]) = (4+\sqrt{-5},1-2\sqrt{-5},9) = (4+\sqrt{-5},1$$
$$-2\sqrt{-5},9,[4+\sqrt{-5}][4-\sqrt{-5}]) = (4+\sqrt{-5},1-2\sqrt{-5},9,21)$$
$$= (4+\sqrt{-5},1-2\sqrt{-5},3) = (4+\sqrt{-5},3) = K_4 = (1-2\sqrt{-5},3).$$

Similarly,

$$(4-\sqrt{-5},1+2\sqrt{-5})=K_3=(4-\sqrt{-5},3)=(1+2\sqrt{-5},3).$$

It follows from the above scheme that associated with the factors

$$(K_1K_2)(K_3K_4)=(K_1K_4)(K_2K_3)=(K_1K_3)(K_2K_4)$$

are the integers

$$7\cdot 3=(1+2\sqrt{-5})(1-2\sqrt{-5})=(4+\sqrt{-5})(4-\sqrt{-5}).$$

Thus the unique factorization of 21 in $R(\sqrt{-5})$ is $K_1K_2K_3K_4$. In the realm of natural numbers the Kummer factors have no objective reality. Hence the name *ideal*.

In the more general[1] quadratic realm $R(\sqrt{m})$, $m\not\equiv 1 \pmod 4$, we have to do with the factorization of integers $x+y\sqrt{m}$. And as above we are led to the consideration of the congruence $x^2-my^2\equiv 0 \pmod p$. If $w^2\equiv m \pmod p$, or $w^2-pr=m$, r an integer, we have a Kummer factor $\{p, w\}$ defined through the congruence $x+wy\equiv 0 \pmod p$. This congruence put in the form of an equation is $x=pz-wy$. From this it follows that $x^2-my^2=p(pz^2-2wzy+ry^2)$. Hence, corresponding to the Kummer factor $\{p, w\}$ of p and $x+y\sqrt{m}$, there is an associated quadratic form (p,w,r), and consequently there exists a class of equivalent forms with determinant m through which p (connected with w as above defined) may be expressed. Then and only then, when the class to which (p, w, r) belongs is a principal class $(1, 0, m)$, can p be expressed through the form $p=x^2-my^2$ $=(x+y\sqrt{m})\ (x-y\sqrt{m})$. In this case and only in this case are the Kummer ideal factors $\{p, w\}$ and $\{p, -w\}$ numbers (algebraic) and have a real existence.

We thus have the condition under which the rules of division that exist in the rational realm are also true in the quadratic realms without the necessity of introducing the Kummer factors, this being evidently the case when the number of non-equivalent classes of quadratic forms with determinant m is unity. When the quadratic form through which the prime ideal p may be expressed is not equivalent to the principal form, it is necessary to introduce the ideal factors to effect uniquely the factorization of the rational prime integers into irreducible factors. Thus it is shown that the theory of quadratic forms with determinant m is exactly correlated with the theory of complex numbers of the realm $R(\sqrt{m})$.

The Kummer theory may with some modification be so changed that the ideal factors of unreal existence may be replaced by ideals of a concrete form. For, if a Kummer ideal prime factor $\{p, w\}$ of p is defined through the congruence $x+wy\equiv 0 \pmod p$ it is seen that the collectivity (complex) of all

[1] See Bachmann, *Allgemeine Arithmetik der Zahlenkörper*, p. 146.

integers of the form $x+y\sqrt{m}$, which are divisible by $\{p, w\}$ with a suitable choice of x, y, may also be expressed through $x=pz-wy$. And that is, the complex of all those algebraic numbers that are divisible by $\{p, w\}$ is of the form $pz+(\sqrt{m}-w)y$. This complex of numbers constitutes what is known as the modul $\mathfrak{a}=[p, \sqrt{m}-w]$. And it is further seen that any number of this modul $pz+(\sqrt{m}-w)y$ when multiplied by any integer of the realm, say $x'+y'\sqrt{m}$ is equal to

$$p(zx' - ryy' + wzy') + (\sqrt{m} - w)(yx' + pzy' - wyy'),$$

if we write $w^2 \cdot m = pr$. Observe that this latter expression is of the form

$$pZ + (\sqrt{m} - w)Y ;$$

and that is, a number of the modul $\mathfrak{a}$ when multiplied by an integer of the realm $R(\sqrt{m})$ is a number of the modul $\mathfrak{a}$. The counterpart of this in the theory of rational integers is: if an integer is divisible by the rational integer a, then the product of the first integer by any other integer is divisible by a. This might in a measure be used to define a rational integer a. It is used by Dedekind to define an ideal $=\mathfrak{i}=[\alpha, \beta]$, where the element p above is replaced by α and where β stands for $\sqrt{m}-w$. Accordingly, the ideal $=\mathfrak{i}$ is defined as the complex of integers $\alpha\lambda+\beta\mu$ where λ and μ run through all the integers of the given realm, and where α and β are definite fixed integers of this realm.

Kummer, in Crelle's Journal, vol. 35 (1846), p. 359, writes as follows regarding the analogy that exists between the theory of complex (algebraic) numbers and chemistry: "Multiplication for the complex numbers corresponds to the chemical compound (Verbindung); to the chemical elements or atomic weights there correspond the prime factors, and the chemical formulae for the decomposition of bodies are precisely the same as the formulae for the factorization of numbers. And the ideal numbers of our theory appear also in chemistry as hypothetical radicals (Radicale) which have not as yet been isolated (dargestellt), but have their reality, as do the ideal numbers in their combination (Zusammensetzung). For example, fluorine[1] which has not yet been isolated (1846) and is still counted as one of the elements, is an analog of an ideal prime factor. Idealism in chemistry however is essentially different from the idealism of complex (algebraic) numbers in that chemical ideal materials when combined with actual materials produce actual compounds: but this is not the case with ideal numbers. Furthermore, in chemistry the materials making up an unknown dissolved (aufgeloesten) body may be tested by means of reagents, which produce precipitates, from which the presence of the various materials may be recognized. Observe that the multiplication of prime ideal numbers produce

[1] Fluorine was isolated in 1886 by Moissant at Paris.

a rational prime integer, so that the reagents of chemistry are the analogues of the prime ideals, these prime ideals being exactly the same as the insoluble precipitate which, after the application of the reagent, settles.

"Also, the conception of equivalence is practically the same in chemistry as in the theory of algebraic numbers. For, as in chemistry, two weights are called equivalent if they can mutually replace each other either for the purpose of neutralization or to bring about the appearance of isomorphism. Similarly, two ideal numbers are equivalent if each of them can make a real rational number out of the same ideal number."

"These analogies which are set forth here are not to be considered as mere 'play things of the mind' in that chemistry as well as that portion of the number theory which is here treated have the same basic concept, namely, that of combination even though within different spheres of being. And from this it follows that those things which are related to this (principle of combination) and the given concepts which necessarily follow with it (that is, with the principle of combination) must be found in both (fields) by similar methods: The chemistry on the one hand of natural materials and on the other hand the chemistry of ideal numbers may both be regarded as the realizations (Verwirklichungen) of the concept of combination and of the concept-spheres dependent upon this (principle of combination) the former to be regarded as a physical which is bound with the accidents of external existence and therefore richer, the latter a mathematical, which in its inner necessity (nothwendigkeit) is perfectly pure, but therefore also poorer than the former."

While the above is not an exact translation, a few phrases having been added to make the meaning clearer, it gives, I believe, the full and exact import of what Kummer wished to convey.

Professor Edward Zeller, the world's authority on Greek philosophy writes:[1]

"Plato's pure mathematics is primarily a preparatory stage of Dialectic (general logic), the number with which it has to do are not ideal but mathematical numbers: not identical with ideas but intermediate between them and the things of sense. Side by side with numbers, the ideas of numbers are also spoken of, but only in the same sense that ideas generally are opposed to things."

And further:

"The more exact distinction between the two kinds of numbers is this; that the mathematical consists of homogeneous unities which can therefore be reckoned together, each with each, whereas with the ideal numbers this is not the case; consequently the former expresses merely quantitative, the latter logical determinations."

It is seen from this that in the notion handed down from Plato addition is an

[1] *Plato and the Older Academy*. By Eduard Zeller. Translated by Alleyne and Goodwin. Page 256.

operation belonging to the mathematical numbers while with the ideal numbers this is not the case. Even with the Greeks it would appear that mathematics was an apparatus for numerical calculation as well as the philosophy of thought.

From recent articles which have to do with the historical development of chemistry and other physical sciences, which have appeared for example in *Science* I am led to believe that the assertion is justified that the lines of thought as well as the laws of thought in mathematics, the sciences and philosophy are today so closely allied with those of earlier times that they are one and the same. Evolution of the human mind has been exceedingly slow. And it is gratifying to note that progress in the entire intellectual structure has been made and is being made in the right direction and along lines where the truth seems of itself to be the guide.

I cannot refrain however from noting certain difficulties which the philosophers seem to experience regarding the *Ideas* and *Numbers* of Plato:

Jowett, in *The Dialogues of Plato*, vol. 2, p. 13 (third edition), writes; "Plato's doctrine of ideas has attained an imaginary clearness and definiteness which is not to be found in his own writings." Jowett then proceeds to enumerate the ways in which these ideas are interpreted and describes the reasons of their misinterpretations and misunderstanding.

Professor Paul Shorey, in *The Unity of Plato's Thought*, Chicago Decennial Publications, first series, vol. 6, p. 82, writes: "Aristotle's account of Plato's later identification of ideas and numbers has been generally accepted since Trendelenburg's dissertation on the subject. Zeller rightly points out that the doctrine is not found in the extant writings, but adds that for Plato numbers are entities intermediate between ideas and things of sense. In my discussion of the subject I tried to establish two points; first that we need not accept the testimony of Aristotle, who often misunderstood Plato and was himself not clear as to the relation of mathematical and other ideas: second, that the doctrine of numbers as intermediate entities is not to be found in Plato, but that the passages which misled Zeller may well have been the source of the whole tradition about ideas and numbers."

It does not seem to me in view of the above quoted passages from Zeller and from what I have given above regarding the Kummer ideal numbers, that there is any incorrectness either in Zeller's interpretation or in that of the equally eminent Plato scholar. In the quoted passage, Zeller does not use the word "entity." Further Shorey (p. 83) writes; "The 'mathematical' numbers then are plainly the abstract, ideal numbers of the philosopher." Surely Kummer and Dedekind were philosophers as well as mathematicians. And, continuing, Professor Shorey writes; "The numbers of the vulgar are concrete numbered things." And this is precisely what I have indicated above. See also Shorey, *Ideas and Numbers Again*, Classical Philology, vol. 22 (1927). What particularly

impresses me is the following, also from Professor Shorey (Proceedings of the Sixth International Congress of Philosophy, p. 578); "And the more closely and critically one studies Plato, the more apparent it becomes that the one way to misunderstand him is to make condescending allowance for the naïveté, the immaturity, the primitive, unscientific, unsophisticated quality of his thought. Mankind has undergone many vicissitudes of experience since Plato wrote, by which philosophers must endeavor to profit. But philosophy itself, metaphysics, epistemology, speculation about ultimates in cosmogony, religion, ethics and politics, and the subtlety and logic of the art of debate have progressed since Plato just as much as epic has improved since Homer, tragedy since Sophocles, sculpture since Phidias, and eloquence since Demosthenes." Happily Professor Shorey does not include the mathematicians in the category of those whose progress by implication has been in the negative direction.

FORCED VIBRATIONS WITH SMALL RESISTANCE, APPROXIMATING RESONANCE[1]

L. W. BLAU, University of Texas

1. Introduction. The general subject of forced vibrations has been discussed rather fully by different authors[2], but the relations derived do not adapt themselves readily to the investigation of certain especially important cases. Some of these are; 1. Approximating resonance, with small resistance; 2. Resonance, with small resistance; 3. Approximating resonance, with zero resistance. In particular, the relation of these three cases to a fourth case; 4. Resonance with zero resistance, has not been clearly set forth. Rayleigh[3] states, in discussing the equation derived by him, that "the change of phase from complete agreement to complete disagreement, which is gradual when friction acts, here" (that is, when there is resonance and the friction is permitted to approach zero) "takes place abruptly. At the same time the amplitude becomes infinite." This statement is indefinite. "At the same time" does not mean *instantaneously*, for it is well known that on solving the differential equation of case 4 the amplitude, that is the maximum of the absolute value of the displacement, does not increase to infinity monotonically, but the relative maxima of this value increase directly as the first

[1] Read before the Texas section of the Mathematical Association of America, Jan. 28, 1928.

[2] Rayleigh, *Theory of Sound*, 2nd edition. The Macmillan company, 1894. Duffing, *Erzwungene Schwingungen Bei Veränderlicher Eigenfrequenz und Ihre Technische Bedeutung*, F. Vieweg u. Sohn, Braunschweig, 1918. Riemann-Weber, *Die Differential-und Integralgleichungen der Mechanik und Physik*, Zweiter, Physikalischer, Teil. F. Vieweg u. Sohn, Braunschweig, 1927, pp. 86 ff.

[3] *Theory of Sound*, 2nd edition. The Macmillan company, 1894, vol. 1, p. 48.

power of the time. From the theory of differential equations it is also known that the solution of (1), below, is a continuous function of the coefficients, so that cases 1, 2, and 3 should pass over into case 4 smoothly. Physical considerations also lead to a similar conclusion, for the impressed force must impart to the system sufficient energy to cause an infinite amplitude, and it is not very plausible that this should be accomplished within one half of a cycle. It is the purpose of this paper to derive an equation for the amplitude of vibration in terms of the frequency of the impressed force, the difference between this frequency and the natural frequency, and the resistance: this relation is well adapted to the investigation of the cases mentioned above. It is found that in cases 1, 2, 3, the first maximum is a maximum maximorum if the initial displacement and the initial velocity are each equal to zero; there is a "beat" frequency the period of which depends upon the difference between the frequency of the impressed force and the natural frequency of the system as well as on the resistance. In passing over to case 4 the above "beat" frequency becomes zero, and the amplitude increases with the first power of the time.

2. The Differential Equation of Motion. Given a system having unit mass and natural frequency of vibration $n/(2\pi)$: let the oscillations of the system be opposed by a force which is directly proportional to the first power of the velocity and let a periodic force of frequency $p/(2\pi)$ be impressed on the system. The differential equation of motion then becomes

$$\ddot{y} + 2R\dot{y} + n^2 y = E \cos pt, \tag{1}$$

where y is the displacement from the position of rest, $\dot{y}$ the velocity, and $\ddot{y}$ the acceleration, while n, R, and E are constants. Let the initial displacement and the initial velocity be each equal to zero; then we have the conditions

$$y(0) = 0 \text{ and } \dot{y}(0) = 0. \tag{2}$$

The general solution of equation (1) is

$$y(t) = A \cos pt + B \sin pt + Ce^{-Rt} \cos (n^2 - R^2)^{1/2} t + De^{-Rt} \sin (n^2 - R^2)^{1/2} t, \tag{3}$$

where

$$C = \frac{(p^2 - n^2)E}{(n^2 - p^2)^2 + 4R^2p^2} \text{ and } D = \frac{-RE(n^2 + p^2)}{(n^2 - R^2)^{1/2}[(n^2 - p^2)^2 + 4R^2p^2]}.$$

In order to satisfy the conditions (2) the constants A and B must have the following values:

$$A = \frac{(n^2 - p^2)E}{(n^2 - p^2)^2 + 4R^2p^2}; \quad B = \frac{2RpE}{(n^2 - p^2)^2 + 4R^2p^2}. \tag{4}$$

Let $(n^2-R^2)^{1/2}=n-\Delta n$, and $n=p+s$: then $n-\Delta n=p+s-\Delta n$. Equation (3) then becomes

$$
\begin{aligned}
y(t) &= A\cos pt + B\sin pt \\
&\quad + e^{-Rt}[C\cos(p+s-\Delta n)t + D\sin(p+s-\Delta n)t] \\
&= [A + e^{-Rt}\{C\cos(s-\Delta n)t + D\sin(s-\Delta n)t\}]\cos pt \\
&\quad + [B + e^{-Rt}\{D\cos(s-\Delta n)t - C\sin(s-\Delta n)t\}]\sin pt.
\end{aligned} \tag{5}
$$

This may be written

$$
y(t) = \Big\{A^2 + B^2 + (C^2+D^2)e^{-2Rt} - 2e^{-Rt}[(A^2+B^2)(C^2+D^2)]^{\frac{1}{2}}\sin\left[(s-\Delta n)t + \tan^{-1}\frac{C}{D} + \tan^{-1}\frac{B}{A}\right]\Big\}^{\frac{1}{2}}\sin(pt+\beta) \tag{6}
$$

where

$$
\beta = \tan^{-1}\frac{A + e^{-Rt}(C^2+D^2)^{1/2}\sin\left[(s-\Delta n)t + \tan^{-1}\frac{C}{D}\right]}{B + e^{-Rt}(C^2+D^2)^{1/2}\cos\left[(s-\Delta n)t + \tan^{-1}\frac{C}{D}\right]}.
$$

Equation (6) represents an oscillatory motion. If we let α represent the amplitude, then

$$
\alpha = \text{maximum}\left|\left(A^2 + B^2 + (C^2+D^2)e^{-2Rt} - 2e^{-Rt}[(A^2+B^2)(C^2+D^2)]^{1/2}\sin(s-\Delta n)t + \tan^{-1}\frac{C}{D} + \tan^{-1}\frac{B}{A}\right)^{1/2}\right|. \tag{7}
$$

It is clear that the amplitudes form an infinte set, defined for values of t where the velocity is zero.

To find the solution of the differential equation satisfying the boundary conditions (2) for the case when $s=0$, $R=0$, and $\Delta n=0$, we let s, Δn, and R each approach zero in equation (6); then we obtain

$$
y(t) = -(1/2p)Et\sin pt, \tag{8}
$$

which is the equation of case (4).

3. Conclusions: We proceed to the interpretation of equations (6), (7), and (8). If s and Δn are not both equal to zero, equation (6) represents an oscillatory motion of varying amplitude. The maxima occur with a frequency $(s-\Delta n)/(2\pi)$, and according to equation (7) the first maximum is the greatest on account of the exponentials. This is quite apparent on sketching the amplitude as given by equation (7). The accompanying figures 1–5 are graphs of this equation for different values of n, p, R and E. In Fig. 1, n was taken

equal to 25, $p=40$, $R=0.3$ and $E=5000$. Thus s is equal to -15. We note from the graph that the first maximum is the greatest, while the first minimum is the smallest. The same features are evident from Fig. 2; here $n=25$, $p=20$, $R=0.3$ and $E=1000$. The "beat" frequency is smaller than in Fig. 1, and the first minimum is larger. In Fig. 3 n, has the same value as before, $p=24$, R is again 0.3, while $E=200$. There are now only three "beats"; the first

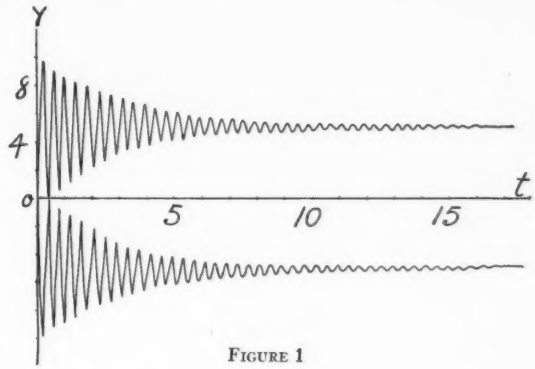

FIGURE 1

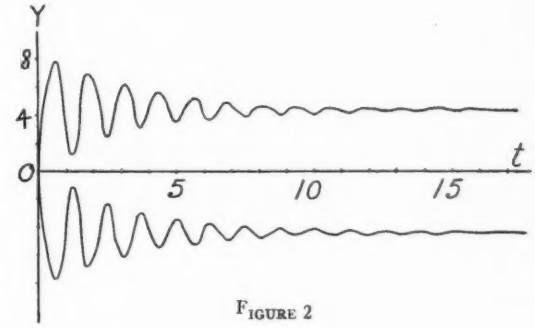

FIGURE 2

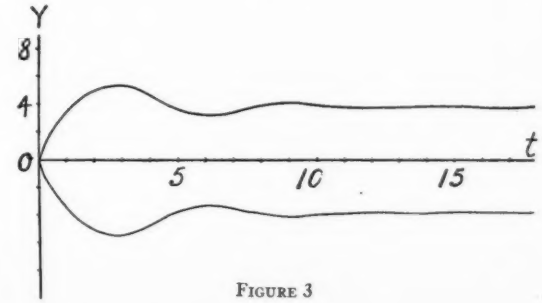

FIGURE 3

minimum is still larger than in Fig. 2. In Fig. 4, $n=p=25$, $R=0.3$ and $E=1000$. Equation (7) shows that even at resonance there is a "beat" frequency of $\Delta n/(2\pi)$. These "beats" are not brought out in this graph, because the frequency n is too large in comparison with R. As R approaches n these "resistance beats" at resonance become more prominent. Fig. 5 shows beats without

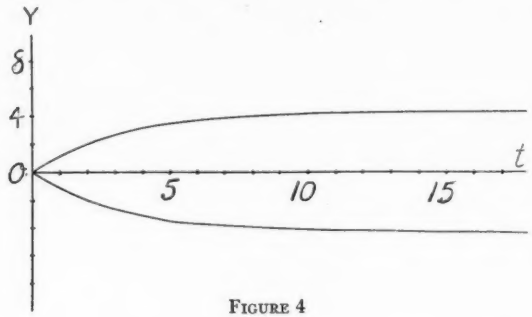

FIGURE 4

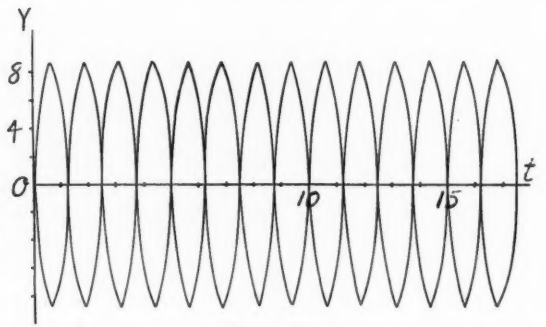

FIGURE 5

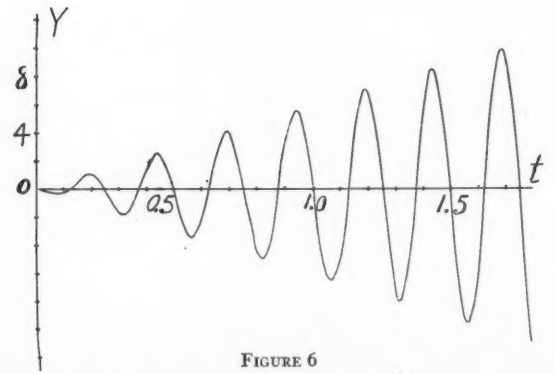

FIGURE 6

damping. In this graph n has the same value as before, $p=20$, and $E=1000$. Fig. 6 is the graph of equation (8).

There is a beat frequency of $(s-\Delta n)/(2\pi)$ for all values of $s \neq \Delta n$. If $s=\Delta n$, that is, if $n^2-p^2=R^2$, the resultant amplitude increases monotonically from zero to the amplitude of the forced vibration in the steady state. This is the only case in which there are no "beats" in a damped system. The difference between the squares of the frequencies may therefore be called the critical frequency difference. An increase in s while Δn remains constant results in an increase in the "beat" frequency. If the system has zero resistance and the impressed force is gradually brought into resonance with the natural frequency of the system, the beat frequency decreases until, when $s=0$ and $\Delta n=0$, the amplitude increases with the first power of the time and becomes infinite with t according to equation (8). In all cases, where the initial displacement and the initial velocity are each equal to zero and where at least one of the quantities s or Δn is different from zero the first relative maximum amplitude is greater than any of the later maxima and greater than the maximum vibrations of the steady motion; for it is apparent from equation (6) that the equation of the steady motion is

$$y(t) = (A^2 + B^2)^{1/2} \sin\left[pt + \tan^{-1}\frac{A}{B}\right]. \tag{9}$$

If the damping is small the steady state is reached much later than if it is large; but in all cases where "beats" are present the first "beat" maximum is greater than any other later maximum vibration.

Equation (6) has been derived from the solution of equation (1) which satisfies the homogeneous boundary conditions. If the boundary conditions are non-homogeneous, the solution of (1) is found by adding to equation (6) the solution of the homogeneous differential equation satisfying the non-homogeneous boundary conditions.

The facts enumerated above show that in any case of forced vibrations the system is subjected to the greatest strain when the first maximum is reached if friction is involved; neglecting fatigue there is little danger of a breakdown after this. This, it may be remarked, confirms our intuitive feeling when standing on a vibrating suspension bridge, that the danger of a collapse has been averted if the bridge has withstood the strain incident to the violent oscillations of the first "beat" maximum.

The writer has made an experimental investigation of these cases; records have been obtained for different values of damping when the frequencies were different and when there was resonance. The results are in excellent agreement with theory.

It is a pleasure to acknowledge my indebtedness to Professor H. J. Ettlinger who suggested the problem.

LINE COMPLEXES IN KINEMATICS

By E. L. REES, University of Kentucky

1. One of the most beautiful applications of the elementary theory of line complexes is found in the theory of the motion of a rigid body in space. Although this subject is treated in several places in the literature it seems to be not very well known and few references are made to it, in spite of the fact that the proofs, both analytic and geometric, are simple and brief.[1]

We shall in this paper develop the theory of linear line complexes and tetrahedral complexes from the point of view of kinematics, using vector methods which are of particular advantage in such work.

2. Central axis. We first show that in the general motion of a rigid body in space the body is instantaneously rotating about and moving in the direction of an axis which is called the *kinematical central axis*.

Let $\boldsymbol{p}$ be the position vector of a point fixed in the body and $\boldsymbol{q}$ the vector from this point to another point also fixed in the body. The position vector of the latter point is $\boldsymbol{r}=\boldsymbol{p}+\boldsymbol{q}$. For points whose velocities have the direction of the angular velocity vector $\boldsymbol{w}$ we have $\boldsymbol{w}\times\dot{\boldsymbol{r}}=\boldsymbol{w}\times\dot{\boldsymbol{p}}+\boldsymbol{w}\times(\boldsymbol{w}\times\boldsymbol{q})=0$, the dots indicating time derivatives. Solving for $\boldsymbol{w}\times\boldsymbol{q}$ we find $\boldsymbol{w}\times\boldsymbol{q}=\boldsymbol{w}^{-1}\times(\boldsymbol{w}\times\dot{\boldsymbol{p}})$, from which $\boldsymbol{q}=\boldsymbol{w}^{-1}\times\dot{\boldsymbol{p}}+u\boldsymbol{w}$, in which u is a scalar parameter and $\boldsymbol{w}^{-1}=\boldsymbol{w}/(\boldsymbol{w}\cdot\boldsymbol{w})$. Hence the locus of points whose velocities have the direction of $\boldsymbol{w}$ is the line $\boldsymbol{r}=\boldsymbol{p}+\boldsymbol{w}^{-1}\times\boldsymbol{p}+u\boldsymbol{w}$, which is therefore the central axis.

3. Definition of linear complex. Before giving the kinematical definition of a linear complex we prove the following theorem:

If a line of a moving body is normal to the velocity of one of its points it is normal to the velocities of all of its points.

PROOF: Let $\boldsymbol{r}=\boldsymbol{p}+u\boldsymbol{q}_1$ be the equation of the line, where $\boldsymbol{p}$ is the position vector of the point whose velocity is normal to the line and $\boldsymbol{q}_1$ is a unit vector having the direction of the line. We have then $\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1=0$. Since $\dot{\boldsymbol{r}}\cdot\boldsymbol{q}_1=\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1+u\boldsymbol{q}_1\cdot\dot{\boldsymbol{p}}_1=0$ for all values of u, it follows that the velocity of any other point of the line is also normal to the line.

All lines of the body which are normal to the velocities of their points constitute a *linear complex*. The *central axis* is called the *axis of the complex*. If the motion is simple rotation the complex consists of all lines which intersect the central axis and is called a *special linear complex*. The complex for transla-

[1] This subject is treated geometrically by A. Mannheim in the Journal de Mathématiques, (2), vol. 11 (1866), pp. 273–279; and by Ch. Brisse in the same journal, vol. 15 (1870), pp. 281–314. References are given here to earlier memoirs on this subject by M. Chasles. Also the subject is treated geometrically, but from a different point of view, by A. Schoenflies in *La Géométrie du Mouvement*, French translation by Ch. Speckel, Gauthier-Villars, Paris, 1893.

tion consists of all lines perpendicular to the direction of motion and is a special linear complex with axis at infinity.

4. Polar system. *All lines of the complex through a point of the body,* being perpendicular to the velocity of this point, form *a plane pencil.*

THEOREM: *There is in general one and only one point of each plane of the body whose velocity is normal to the plane.*

To prove this theorem let $\boldsymbol{a}\cdot\boldsymbol{q}=k$ be the equation of the plane, the origin being at the tip of $\boldsymbol{p}$. If the velocity of the point $\boldsymbol{p}+\boldsymbol{q}$ is normal to the plane, we have $\boldsymbol{a}\times(\dot{\boldsymbol{p}}+\boldsymbol{w}\times\boldsymbol{q})=0$, and therefore $\boldsymbol{w}\cdot\boldsymbol{aq}=k\boldsymbol{w}+\boldsymbol{a}\times\dot{\boldsymbol{p}}$, which in general defines one and only one point. If the plane is parallel to the central axis the point is at infinity. If the motion is simple rotation and the plane contains the axis then every point of the plane satisfies the condition.

The plane through a point which is perpendicular to the velocity of this point is called the *polar plane* of the point and the point is called the *pole* of the plane. We have as corollaries:

All lines of a plane through its pole belong to the complex, and conversely, all lines of the complex in the plane pass through the pole.

The polar planes of points of the central axis are normal to this axis.

5. Conjugate lines. If the equation of a line l of the body is $\boldsymbol{r}=\boldsymbol{p}+u\boldsymbol{q}_1$, then the equation of the polar plane of any point of l is

$$[\boldsymbol{r}-(\boldsymbol{p}+u\boldsymbol{q}_1)]\cdot(\dot{\boldsymbol{p}}+u\dot{\boldsymbol{q}}_1)=0, \text{ or } \boldsymbol{r}\cdot\dot{\boldsymbol{p}}-\boldsymbol{p}\cdot\dot{\boldsymbol{p}}+u(\boldsymbol{r}\cdot\dot{\boldsymbol{q}}_1-\boldsymbol{p}\cdot\dot{\boldsymbol{q}}_1-\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1)=0.$$

Therefore all such polar planes pass through the intersection of the planes $\boldsymbol{r}\cdot\dot{\boldsymbol{p}}=\boldsymbol{p}\cdot\dot{\boldsymbol{p}}$ and $\boldsymbol{r}\cdot\dot{\boldsymbol{q}}_1=\boldsymbol{p}\cdot\dot{\boldsymbol{q}}_1+\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1$. The equation of this line of intersection, which is the axis of the pencil or polar planes corresponding to the points of l regarded as poles, is $\boldsymbol{r}=m\dot{\boldsymbol{p}}+n\dot{\boldsymbol{q}}_1+v\dot{\boldsymbol{p}}\times\dot{\boldsymbol{q}}_1$, where m and n are determined by the equations

$$\dot{\boldsymbol{p}}^2m+\dot{\boldsymbol{p}}\cdot\dot{\boldsymbol{q}}_1n=\boldsymbol{p}\cdot\dot{\boldsymbol{p}} \text{ and } \dot{\boldsymbol{p}}\cdot\dot{\boldsymbol{q}}_1m+\dot{\boldsymbol{q}}_1^2n=\boldsymbol{p}\cdot\dot{\boldsymbol{q}}_1+\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1.$$

It is easy to show that the polar planes of the points of l', the axis of the pencil, form another pencil with l as axis. The lines l and l' are called *conjugate lines*. Thus the conjugate of a line may be regarded as the axis of the pencil of polar planes of points of the line or as the locus of the poles of the planes through the line.

If l is parallel to the central axis, that is, if $\boldsymbol{w}\times\boldsymbol{q}_1=0$, and therefore $\dot{\boldsymbol{q}}_1=0$, the conjugate l' is at infinity, for then one of the planes which determine l' is at infinity. Conversely, the conjugate of a line at infinity is parallel to the central axis. It is the locus of the poles of a system of parallel planes and is called a *diameter of the complex*. Hence *all diameters are parallel to the central axis.*

By simple arguments we may prove the following:

All lines of the complex are self conjugate.

All lines intersecting a pair of conjugate lines belong to the complex, and if a line of the complex intersects one of two conjugates it intersects the other also.

We now prove the theorem:

The common perpendicular of two conjugate lines intersects the central axis perpendicularly.

Let $\boldsymbol{b}$ be a vector perpendicular to l and l', then $\boldsymbol{b}\cdot\boldsymbol{q}_1=0$, and $\boldsymbol{b}\cdot\dot{\boldsymbol{p}}\times\dot{\boldsymbol{q}}_1=0$ or $\boldsymbol{b}\cdot\dot{\boldsymbol{p}}\times(\boldsymbol{w}\times\boldsymbol{q}_1)=0$. Expanding this last equation, we get $\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1\boldsymbol{w}\cdot\boldsymbol{b}-\dot{\boldsymbol{p}}\cdot\boldsymbol{w}\boldsymbol{b}\cdot\boldsymbol{q}_1=0$, or in view of the first equation, we have $\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1\boldsymbol{w}\cdot\boldsymbol{b}=0$. Either $\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1=0$ and the line belongs to the complex and is therefore self conjugate, or $\boldsymbol{w}\cdot\boldsymbol{b}=0$ and $\boldsymbol{b}$ is perpendicular to the central axis. This common perpendicular is accordingly a line of the polar plane of a point of this central axis; and being a line of the complex it must intersect the axis in the pole of this plane.

6. Motion of a plane. We next consider certain theorems concerning the motion of a plane of the body.

The locus of points of a plane whose velocities lie in the plane is in general a straight line; and this line is the characteristic of the plane.

The equation of a plane may be written $\boldsymbol{r}=\boldsymbol{p}+u\dot{\boldsymbol{q}}_1+v\boldsymbol{q}_1\times\dot{\boldsymbol{q}}_1$, where $\boldsymbol{p}$ is a point of the plane and $\boldsymbol{q}_1$ a vector normal to the plane.[1] For points whose velocities are in the plane, we have $\dot{\boldsymbol{r}}\cdot\boldsymbol{q}_1=\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1+u\boldsymbol{q}_1\cdot\ddot{\boldsymbol{q}}_1=0$. Since $\boldsymbol{q}_1\cdot\ddot{\boldsymbol{q}}_1=-\dot{q}_1^2\neq 0$, this equation determines a unique value of u which, when substituted in the equation of the plane, gives a line of that plane for the locus of points whose velocities are in the plane, as was to be proved. If $\boldsymbol{p}$ is a point of this line, $u=0$ and the equation of the line is $\boldsymbol{r}=\boldsymbol{p}+v\boldsymbol{q}_1\times\dot{\boldsymbol{q}}_1$.

The equation of the characteristic[2] of this plane is $\boldsymbol{r}=\boldsymbol{p}+\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1\dot{q}_1^{-1}+v\boldsymbol{q}_1\times\dot{\boldsymbol{q}}_1$; and if $\boldsymbol{p}$ is a point with velocity in the plane then $\boldsymbol{r}=\boldsymbol{p}+v\boldsymbol{q}_1\times\dot{\boldsymbol{q}}_1$. Hence the theorem.

The direction of the conjugate of the characteristic is determined by $\dot{\boldsymbol{p}}\times\mathrm{t}(\boldsymbol{q}_1\times\ddot{\boldsymbol{q}}_1)$, that is, by $\boldsymbol{q}_1$, which shows that it is normal to the plane and therefore to the characteristic.

By a simple argument it may be shown that the point at which the conjugate intersects the plane is the pole of the plane and that *the velocity of the point of the plane which lies at the foot of the perpendicular from the pole to the characteristic is in the direction of the characteristic.*

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[1] We assume in this discussion that $\dot{q}_1=\neq 0$. If $\dot{q}_1=0$ then $W\times\boldsymbol{q}_1=0$ and the plane is rotating instantaneously about an axis perpendicular to it or there is no rotation at all. In this case all of the points of the plane would have velocities in the plane or there would be no such points.

[2] We use the term *characteristic of a plane* in the usual sense to mean the ultimate intersection of neighboring planes of a one parameter family. Here the successive positions of a plane of the body constitute a one parameter family. The characteristic is found by the usual method to be the intersection of the planes $(\boldsymbol{r}-\boldsymbol{p})\cdot\boldsymbol{q}_1=0$ and $(\boldsymbol{r}-\boldsymbol{p})\cdot\dot{\boldsymbol{q}}_1-\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1=0$, which is the line $\boldsymbol{r}=\boldsymbol{p}+\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1\dot{q}_1^{-1}+v\boldsymbol{q}_1\times\dot{\boldsymbol{q}}_1$.

This relation is a reciprocal one, the conjugate line being the characteristic of the plane through it which is perpendicular to the characteristic of the original plane. Indeed either of two perpendicular conjugates is the characteristic of a plane perpendicular to the other and the pole of this plane is the point in which the other conjugate pierces it.

7. The tetrahedral complex. The lines determined by the points of the body, and their velocities at a given instant, form a tetrahedral complex. In view of the preceding article it follows from this definition that *the characteristics of the planes of the body form a tetrahedral complex.*

8. Properties of tetrahedral complexes. Since the direction of the characteristic of a plane is determined by $q_1 \times \dot{q}_1 = w - w \cdot q_1 q_1$, which is the component of w parallel to the plane, it may be shown geometrically that *the lines of a tetrahedral complex which pass through a point form an orthogonal cone*; also it is easy to see that *the lines of the complex which lie in a plane envelop a parabola.*

The axes of the two pencils of orthogonal planes, whose lines of intersection are the elements of the orthogonal cone, are the tangent to the trajectory of the vertex of the cone and a line through the vertex parallel to the central axis.

The point of the characteristic of a plane whose velocity has the direction of the characteristic is the foot of the perpendicular from the pole of the plane to the characteristic. This point is also the vertex of the parabolic envelope of the lines of the complex in the plane.

The focus of the parabolic envelope of lines of the complex in a plane is the pole of the plane with respect to the linear complex. If the intersection of two orthogonal planes is the axis of the parabolic envelope of one of the planes it is also the axis of the envelope of the other plane. The vertex of each parabola lies on the focus of the other. In other words the two envelopes are focal parabolas.

9. Relations between the Linear Complex and the Tetrahedral Complex. Below are given a few of the more important relations between the lines of the linear complex and those of the tetrahedral complex.

The lines of the tetrahedral complex which are determined by the velocities of the points of a given line are perpendicular to the conjugate of this line with respect to the linear complex.

A pair of orthogonal conjugate lines with respect to the linear complex are lines of the tetrahedral complex; and conversely, any line of the tetrahedral complex is one of a pair of orthogonal conjugate lines with respect to the linear complex.

These statements may be proved by simple geometric arguments.

AMERICAN STANDARD MATHEMATICAL SYMBOLS

Prepared by the SECTIONAL COMMITTEE ON SCIENTIFIC AND ENGINEERING SYMBOLS AND ABBREVIATIONS

Foreword

The Sectional Committee on Scientific and Engineering Symbols and Abbreviations was organized under the procedure of the American Engineering Standards Committee as the culmination of a series of efforts on the part of the A.S.C.E., A.S.T.M., A.I.E.E., S.P.E.E., A.S.M.E., and other national organizations. The societies which accepted joint sponsorship are the A.A.A.S., A.S.C.E., A.I.E.E., S.P.E.E., and the A.S.M.E. This Sectional Committee consisting of representatives of thirty-four national societies, associations, and government departments held its organization meeting on January 21, 1926, and Subcommittee No. 6 was appointed to report upon mathematical symbols.

This Subcommittee held its organization meeting May 1, 1926. Its report was submitted to the Sectional Committee under date of January 22, 1927. After approval by the Sectional Committee the report was submitted to the sponsors under date of June 3, 1927. In due course approval was voted by all five sponsors.

After a review of the report in the light of certain criticisms not of a fundamental nature which had been brought out in the course of securing sponsorial approval the Subcommittee reaffirmed the report as originally drafted with the exception of a few minor revisions of an editorial character.

The proposed standard was submitted under date of December 5, 1927, to the American Engineering Standards Committee for approval and on January 11, 1928, the sponsors were notified of its approval as an "American Standard."

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MATHEMATICAL SYMBOLS

Arithmetic and Algebra

1. $=\ \neq\ +\ -\ \pm\ \mp\ <\ >\ \leqq\ \geqq$ () [] % $\infty\ \approx$ (for approximately equal to)
2. $a\times b=a\cdot b=ab$; $a\div b=a/b=\frac{a}{b}$ (Influence extends to next + or −). Thus, $a-b/c-d$ should not be used for $(a-b)/(c-d)$. Note that $\frac{a}{b}$ is difficult to print in running text.
3. $a/b=c/d$ for proportion. Discourage $a:b::c:d$.
4. Notation by powers of 10 for very large or very small numbers is recommended: as 3.140×10^{9} and 3.140×10^{-6}. The notation $0.0^{5}314$ is useful in tables, to indicate that there are five zeros after the decimal point.
5. In writing numbers having a larger number of digits, half-spaces may well be used instead of commas to separate groups of digits. In writing decimals, the 0 before the decimal point should not be omitted (except in tables).
6. $|x|$ = absolute value of x. $x!=1\cdot2\cdot3\cdots x$. Discourage $\underline{|x}$.
7. $\sqrt{x}=+\sqrt{x}$, not $\pm\sqrt{x}$ (x being real and positive). $a^{1/n}=\sqrt[n]{a}$. $a^{-n}=1/a^{n}$. $\exp x=e^{x}$ is useful when x is a complicated expression. Note that the bar or vinculum after the $\sqrt{}$ is very expensive to print.
8. When $\log x$ is ambiguous, use $\log_{10}x$ or $\log_{e}x$. The notation $\ln x$ may be mentioned as an alternative for $\log_{e}x$.
9. $P(n, r)=n(n-1)(n-2)\cdots(n-r+1)$
   $C(n, r)=[n(n-1)(n-2)\cdots(n-r+1)]/[1\cdot2\cdot3\cdots r)$ = binomial coefficients.

A common alternative for $C(n, r)$ is $\binom{n}{r}$; this however is difficult to print in running text.

10. $a \propto b$ (meaning a varies directly as b).

Elementary Geometry

11. $\angle$ Δ $\parallel$ $\perp$ $\odot$ $\square$ $\therefore$

Analytic Geometry

12. x, y, z; ξ, η, ζ; rectangular coordinates. Right-handed system preferred.
13. ρ, s = intrinsic coordinates. ρ = radius of curvature, s = length of arc.
14. $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$, direction cosines.
15. r, θ = polar coordinates. ψ = angle from radius vector to tangent.
16. r, θ, ϕ = spherical coordinates. θ = co-latitude, ϕ = longitude. (Usage general in mathematical physics; other notations are used in astronomy.)
17. r, θ, z = cylindrical coordinates. (Usage diverse.)
18. Conics: e = eccentricity. p = semi-latus rectum (usage general in U.S.).
19. Straight line; $y = mx + b$.

Trigonometric and Hyperbolic Functions

20. $^\circ$ $'$ $''$ $\sin x$, $\cos x$, $\tan x$, $\operatorname{ctn} x$, $\sec x$, $\csc x$.
21. $\sin^{-1}x$ = the principal value of the angle whose sine is x (when x is real). Thus, $-\pi/2 \leqq \sin^{-1}x \leqq \pi/2$, $0 \leqq \cos^{-1}x \leqq \pi$, $-\pi/2 \leqq \tan^{-1}x \leqq \pi/2$. (Discourage arc sin x).
22. $\sin^2 x$ for $(\sin x)^2$ is an exceptional notation, justified by usage. Similarly for $\cos^2 x$, etc.
23. $\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{ctnh} x$, $\operatorname{sech} x$, $\operatorname{csch} x$.
24. $\cosh^{-1}x$ = the principal value (when x is real). (Discourage arc sinh x).
25. $\sinh^2 x$ for $(\sinh x)^2$ is an exceptional notation, justified by usage. Similarly, $\cosh^2 x$, etc.
26. In general f^{-1} means the inverse of the function f; and f^2 denotes iteration of the functional operation. But in exceptional cases, f^2 may denote the square of the function f (as in $\sin^2 x$ and $\sinh^2 x$). In general, $[f(x)]^{-1} = 1/f(x)$.

Calculus, etc.

27. If $y = f(x)$, derivative $= y' = f'(x) = \dfrac{dy}{dx} = D_x y$.

Second derivative $= y'' = f''(x) = \dfrac{d(y')}{dx} = D_x^2 y = \dfrac{d^2y}{dx^2}$. Note: $\dfrac{d^2y}{dx^2}$ cannot be regarded as a fraction, except when x is the independent variable; in general, $\dfrac{d^2}{dx^2} = D_x^2$ is a symbol of operation on y.

28. If $u=f(x, y)$, partial derivative $=u_x=f_x(x, y)=D_x u=\dfrac{\partial u}{\partial x}$. Similarly, $u_{xy}=f_{xy}(x, y)=D_y(D_x u)=\dfrac{\partial^2 u}{\partial y \partial x}$. Note: $\dfrac{\partial^2 u}{\partial y \partial x}$ and $\dfrac{\partial u}{\partial x}$ are not fractions; $\dfrac{\partial}{\partial x}=D_x$ and $\dfrac{\partial^2}{\partial y \partial x}=D_y D_x$ are symbols of operation.

29. $\Delta y=$ increment, $dy=$ differential. $\delta y=$ variation. $\sum_{i=a}^{b}=$ summation.

30. $\dot{x}=dx/dt=v$; $\ddot{x}=dv/dt$ (used only for differentiation with respect to the time t, and difficult to print).

31. $\lim_{x \to a}(y)=b$; $y \to b$ as $x \to a$. (Discourage $\doteq$.)

32. $\int_a^b f(x)dx$. $F(x)\,|_a^b = F(b)-F(a)$. $\int \int f(x,y)dxdy=\int [\int f(x,y)dx]dy$.

33. $\pi=3.1416\ldots e=2.718\ldots i$ (or j) $=\sqrt{-1}$.

34. If $z=x+iy$, then $|z|=$ absolute value, or magnitude, $<z=$ angle, $R(z)$ and $I(z)=$ real and imaginary parts, $\bar{z}=$ conjugate of z. (Where $\bar{z}$ is difficult to print, use conj z.)

Special Functions

35. Bessel Functions. The notation used in G.N. Watson's Treatise, 1922, as endorsed by E.P. Adams in the Smithsonian Tables, 1922, is recommended.

36. Bernoulli numbers. Of the five or six different notations in use, the notation $B_1, B_3, B_5, \cdots$ has historical priority and many practical advantages, but the notation $B_1, B_2, B_3, \cdots$ is the one most used in recent years. To indicate what usage is being followed, authors will do well to state explicitly the value of the first few numbers, as $B_1=1/6$, $B_2=1/30$, $B_3=1/42, \cdots$

37. $\gamma=0.5772\ldots$ (Euler's constant.)

Vector Analysis[1]

38. Vectors to be indicated in printed matter by letters in bold face type, and in written manuscript by letters modified by a bar above (or by the doubling of some part of the character). The magnitude of a vector to be indicated in print by the corresponding italic letter, and in manuscript (when necessary) by the use of the absolute value signs, $|\ |$.

39. The scalar product, or dot product, $=\boldsymbol{a}\cdot\boldsymbol{b}$, the dot being centered. (Other notations are $S\boldsymbol{ab}$, or $(\boldsymbol{ab})$ in round parentheses.)

40. The vector product, or cross product, $=\boldsymbol{a}\times\boldsymbol{b}$, the cross being small. (Other notations are $V\boldsymbol{ab}$, or $[\boldsymbol{ab}]$ in square brackets.)

41. $i, j, k=$ unit vectors along the axes (right-handed system).

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[1] NOTE: As to further questions of notation in Vector Analysis (including Tensor Analysis), the desirability of a thorough-going attempt to bring uniformity out of the present diversity of usage is recognized, and the appointment of a special committee to take up this subject has been recommended.

Abbreviations

42. It is desirable to distinguish between (1) a "symbol," that is, a single letter or single letter affected with subscripts, etc., which is to be used to represent a numerical value in a formula; and (2) an "abbreviation," which may consist of several letters, but is not intended to be substituted for a numerical quantity in a formula.

43. Abbreviations such as ft/sec^2, ft-lb/min, etc., should not be further condensed, lest clearness be sacrificed to brevity.

NOTE: The recommendations concerning terms and symbols in elementary mathematics contained in Chapter 8 of the report on the Re-organization of Mathematics in Secondary Education, made in 1923 by the National Committee on Mathematical Requirements (under the auspices of the Mathematical Association of America and reprinted by Houghton Mifflin Company in 1927), are, with one or two important exceptions, endorsed.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. INTEGRAL BASES

By E. T. BELL, California Institute of Technology

Under the above title a review of W. E. H. Berwick's recent Cambridge Tract on this topic appeared in the Monthly for March, 1928, pp. 142–3. The following remarks are not intended in any way as a criticism either of the tract or of the review, both of which are admirable from the respective points of view from which they were written. It is important, however, to call attention to a significant, highly practical paper on this subject which is not mentioned either by the author of the tract or by the reviewer. (The author's tract went to press before the paper appeared). Students who are interested in these subjects will find it profitable to master the paper by Professor N. R. Wilson in the Transactions of the American Mathematical Society, vol. 29 (1927), pp. 111–126.

The minimum of space in which Wilson disposes of Woronoi's cubic problem will commend itself to all who have ploughed through the Russian original or Sommer's abstract of the same. Further, Wilson's methods apply at once to the quintic field, and, with but slight extensions, settle the points in question for such fields and for all higher fields.

There is another item of importance, not considered by the reviewer. In his tract, Berwick bases his account of Zolotareff's theory (apparently) on the

German abstract (including Bachmann's, Zahlentheorie, vol. 5). On reading Berwick's remarks, one gets the impression, perhaps not intended by the author, that Zolotareff's theory is a rank failure. This is unfortunate, as the facts are otherwise. Zolotareff's theory should be rescued from oblivion in the form in which Zolotareff himself put it forth. The theory is sound. I wo uld hesitate to make this assertion were it not reinforced by an opinion which all arithmeticians must receive with respect—namely that of Professor J. V. Ouspensky. The fact, independent of any mere opinion, seems to be thus: the accounts in Bachmann and the Fortschritte are based on insufficient knowledge of the Russian original. On other grounds (postulational formulations) also I believe that the current estimate of Zolotareff's theory is erroneous. At worst, a slight refinement of statement is all that is required to make it unassailable.

It is time that Zolotareff's work, with an adequate critical commentary, was made available to those who do not read Russian.

II. Beginning Geometry and College Entrance

By H. W. Tyler, Massachusetts Institute of Technology

In a recent interesting letter to the Monthly,[1] Professor Beatley has cogently urged the appropriateness of including a certain moderate amount of three-dimensional work in connection with the present requirement in plane geometry, in view of the fact that a larger proportion of the students concerned will have no other systematic study of solid geometry. Sympathizing decidedly with this idea, I have been from the standpoint of a scientific institution much concerned with the probable reaction on the solid geometry in case a substantial portion of it should have been anticipated as a part of the more elementary subject. The logical remedy for such duplication would naturally be the division of elementary geometry as a whole on a more fundamental basis than that traditionally expressed by the titles Plane and Solid. The division ought, it seems to me, to be based largely on relative difficulty and relative importance. Geometry A would, accordingly, include those parts of present plane geometry which are deemed important for boys and girls who expect to attend arts colleges or who are taking the school course without reference to more advanced study. It would contain also the simpler and more frequently used theorems of solid geometry, with special attention to mensuration. Geometry B, would then consist of the principal matters left over from the present plane and solid geometry so far as they are of value for the future student of science, with a large proportion of original exercises in both two and three dimensions. This course

[1] Ralph Beatley, *Beginning Geometry and College Entrance*, this Monthly, vol. 35 (1928), pp. 80–83.

would be taken by boys and girls of superior mathematical capacity and ordinarily required of candidates for admission to scientific institutions.

Such a redistribution of material would naturally carry with it the probability of a serious examination of the whole field of elementary geometry and would signify something more than a mere permutation of present elements and textbook pages. It would need the careful study of a commission, representing both schools and colleges, and would lead presumably to such a progressive modification of College Board examinations as Professor Beatley has suggested.

III. Remark on the Note "Some Geometric Illustrations for the Elementary Course in Differential Equations"[1]

By J. D. Tamarkin, Brown University

It was not until recently that I became aware of the fact that property (A) (p. 27) had been stated and used for graphical solution of linear differential equations of the first order by E. Czuber, *Beitrag zum graphischen Integration der linearen Differentialgleichungen erster Ordnung*, Zeitschrift für Mathematik und Physik, 44 (1899), pp. 41–49. The proof is different from ours.

Czuber's method has been extended to some non-linear equations in an interesting paper by T. Kojima, *On graphical solutions of some differential equations of the first order*, The Science Reports of the Tôhoku Imperial University, (1), vol. 3, (1914), pp. 289–302.

IV. A Note on the Linear Differential Equation of the First Order

By E. G. Olds, Carnegie Institute of Technology

In the January, 1928, Monthly[2], Mr. J. D. Tamarkin gives *Some Geometric Illustrations for the Elementary Course in Differential Equations*. In the case of $dy/dx+p(x)y+q(x)=0$ he shows that its family of integral curves possess the property: (*A*) *The tangents at the points of intersection of the family of integral curves with any line parallel to the y-axis are either concurrent or parallel.*

Another proof of the above property is as follows: Let the line, $x=a$, cut the integral curves C_1, C_2, C_3, in points, the ordinates of which are y_1, y_2, y_3, respectively. Also set $q(a)=Q$, and $p(a)=P$.

Then the slopes of the tangents to the integral curves at $x=a$ are, respectively, $y_i' = -Qy_i - P$ $(i=1, 2, 3)$ and, for $Q=0$, the slopes become equal to $-P$ and the lines are parallel.

In general, the equations of the tangents are:

$$y - y_i = y_i'(x - a) \text{ or, in another form, } y_i' x - y + (y_i - ay_i') = 0.$$

[1] This Monthly, vol. 35 (1928), pp. 27–29.

[2] This Monthly, vol. 35 (1928), pp. 27-29.

The condition that these lines be concurrent is given by the formula

$$\begin{vmatrix} y_1' & 1 & y_1 - ay_1' \\ y_2' & 1 & y_2 - ay_2' \\ y_3' & 1 & y_3 - ay_3' \end{vmatrix} = 0,$$

which is readily verified.

V. An Extension of the Gauss Problem of Eight Queens

By Leonard M. Blumenthal, Johns Hopkins University

1. Introduction. The problem of ascertaining the number of ways in which eight queens may be placed on a chess board so that no queen can capture any other was proposed to Gauss by Nauck.[1] In a correspondence with his pupil, Schumacher, Gauss, "after finding the number to be 76 and then 72, ultimately arrived at 92, which has since been recognized as the correct solution.[2]" The purpose of this paper is to consider an extension of the problem to the case of n queens on a board of n^2 squares. It is shown that whenever n is not a multiple of two or three, many solutions may readily be obtained. Further, the problem is seen to be equivalent to one in combinatory analysis[3].

2. If the determinant $|a_{ij}|$ $(i, j = 1, 2, \cdots, n)$ be expanded and all terms rejected in which two elements have either the sum or the difference of their subscripts identical, then the remaining terms will give all the solutions of the problem.[4] A moment's inspection justifies the rule, since each term involves but one element from each row and column (thus preventing castle-wise capture) and the rejected terms exclude the cases in which two or more queens lie in the same diagonal.

This method of solution is of practical value only for $n = 3, 4, 5$; for the board of 64 squares, for example, it would involve calculation of 20160 terms. The method does give rise, however, to the following very suggestive statement of the problem:

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[1] For a history of the problem see S. Günther, *Zur mathematischen Theorie des Schachbretts*, Grunert's Archiv der Mathematik und Physik, vol. 56, part 3, pp. 281–292. See also W. W. R. Ball, *Mathematical Recreations and Essays*, 7th edition pp. 113–118.

[2] J. W. L. Glaisher, *On the Problem of Eight Queens*, Edin. Phil. Mag., vol. 48, pp. 457–467.

[3] Quite another method of approach is given by W. H. Bussey, *A Note on the Problem of the Eight Queens*, this Monthly, vol. 29 (1922), p. 252. This interesting note shows "that in the special case in which n is a prime number p there is a connection between the problem of the queens and the lines of the finite plane geometry of p points to the line." Prof. Bussey obtains by use of finite geometry methods $p^2 - 3p$ of the total number of solutions. Only for $p = 5$ does this method give all of the solutions of the problem.

[4] This method, due to Günther and developed somewhat by Glaisher in the paper referred to, is rendered simpler of application and more suggestive with the notation adopted here.

Given two sets of numbers $(i, j = 1, 2, 3, \cdots, n)$, *to select n pairs, exhausting the two sets, so that no two pairs will have identical sums or differences.*

Using each pair as the subscripts of an element, the n pairs thus selected will furnish a solution, and the total number of ways in which the n pairs may be chosen will be the totality of solutions.

3. Consider the pairing $(n \neq 2k)$

$$(1) \qquad \begin{pmatrix} 1,2,3,\cdots, & \frac{1}{2}(n+1), & \frac{1}{2}(n+1)+1, \cdots, & n \\ 1,3,5,\cdots, & n, & 2, & \cdots, n-1 \end{pmatrix}$$

where a number i in the first row has its j directly beneath it. We obtain for $i+j$ the two sequences

$$(a) \qquad 2,5,8,\cdots, \tfrac{1}{2}(3n+1)$$

$$(b) \qquad \tfrac{1}{2}(n+1)+3, \tfrac{1}{2}(n+1)+6, \cdots, 2n-1$$

Evidently no number of sequence (b) will be contained in sequence (a) if the first term of (b) is not contained in (a). Now the numbers of (a) are all congruent to 2 (mod 3), whence if $\frac{1}{2}(n+1)+3$ is in (a) we must have

$$\tfrac{1}{2}(n+1)+3 \equiv 2 \pmod 3 \text{ or } n = 3k.$$

It is easily shown[1] that if $n = 3k$, then $\frac{1}{2}(n+1)+3$ will be contained in (a).

The differences $i-j$ form the two sequences

$$0, -1, -2, \cdots, -\tfrac{1}{2}(n-1)\ ;\ 1,2,\cdots, \tfrac{1}{2}(n-1)$$

and hence no two of these are identical. Thus we have that (1) is a *solution for every n not a multiple of two or three*, and for *only* such n's.

4. Solutions may be obtained from a given solution by the following substitutions:

(1) $\binom{i,\ j}{j,\ i}$, reflection in principal diagonal.
(2) $\binom{i,\ j}{n-j+1,\ n-i+1}$, reflection in secondary diagonal.
(3) $\binom{i,\ j}{i,\ n-j+1}$, reflection in middle column.
(4) $\binom{i,\ j}{n-i+1,\ j}$, reflection in middle row.
(5) $\binom{i,\ j}{j,\ n-i+1}$, rotation 90 degrees clockwise.

Carrying out the square and the cube of (5) rotates a solution 180 and 270 degrees respectively. The solutions obtained by the above substitutions are not, of course, all distinct: e.g., the product of (3) and (4) is the square of (5).

A method of obtaining $(n-1)$ *distinct* solutions from a given one is to apply the substitution $\binom{i,\ j}{i,\ j+1}$ $(n-1)$ times to the given solution, the subscripts to be

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[1] Let $n = 3k$ and represent by

$$3j-1 \quad (j = 1, 2, \cdots, \tfrac{1}{2}(3k+1))$$

the numbers of sequence (a). Then $3j-1 = \frac{1}{2}(3k+1)+3$ for $j = \frac{1}{2}(k+3)$ which is among the values j assumes; in fact, the number $\frac{1}{2}(3k+1)+3$ will be the $\frac{1}{2}(k+3)$rd number of sequence (a).

reduced, mod n. The solutions will be distinct whenever the method furnishes them. To secure criteria to determine when the substitution yields solutions let

$$(A) \qquad \begin{pmatrix} 1,2,3, \cdots, j, \cdots, n \\ i_1, i_2, i_3, \cdots, n, \cdots, i_n \end{pmatrix}$$

be a solution, and consider the result of the substitution, i.e.,

$$(B) \qquad \begin{pmatrix} 1, & 2, & 3, \cdots, j, \cdots, & n \\ i_1+1, & i_2+1, & i_3+1, \cdots, 1, \cdots, & i_n+1 \end{pmatrix}.$$

Applying the test developed in section 2, we have that (B) will be a solution if and only if

$$j \neq k + i_k; \quad j \neq k - i_k \qquad (k = 1,2, \cdots, n).$$

This furnishes a very practical device in testing for a solution when n is large.

In conclusion, we remark that the above methods yield all of the 10 solutions for the case $n=5$ with a minimum of labor; that 28 of the 40 solutions for the case $n=7$ are obtained with the same ease (10 more solutions being immediately forthcoming upon securing two solutions that are not of the type treated here). More than 88 solutions for the case $n=11$ may be written down at once.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Wave Mechanics; An Introductory Sketch. By H. F. BIGGS. Oxford University Press, American Branch, New York, 1927; 77 pages. Price $1.50.

In the words of the author, "This sketch is written for physicists of ordinary mathematical attainments, especially those to whom the language difficulty would make the reading of Schrödinger's original papers too laborious for a first plunge."

Wave-mechanics had its origin in the attempt by de Broglie to reconcile the undulatory theory of light with the apparent necessity of ascribing atomic properties to a field of radiation in order to account for the phenomena associated with the photoelectric effect. The hypothesis which de Broglie makes at the outset is that every mass particle, in the older sense, is the centre of a pulsation in space; to be consistent with the requirements of the restricted theory of relativity, the phase velocity, u, must be connected with the speed of the

particle, v, by the relation $uv = c^2$, where c is the velocity of light. One at once asks what it is that is pulsating, or through what are the pulsations transmitted? No answer can be given to these questions; our customary concepts are inadequate to picture a model that works in accordance with the requirements of such a theory. Schrödinger, however, showed that it is possible to relate the amplitude of these hypothetical pulsations to the electrical density in such a way as to lead to satisfactory results in a purely formal manner.

In the first chapter of this book, the author gives an excellent description of de Broglie's theory, and shows how it leads to Schrödinger's differential equation. This equation is then applied to the hydrogen atom; and the appropriate solution is stated without giving the calculation in detail. It may be noted that Schrödinger's original solution of this problem can be very much simplified by using the Laguerre polynomials. This solution is then shown to lead to the same spectral series as are given by Bohr's theory. *Wave-mechanics*, however, goes further in that it determines the intensities of spectral lines, as well as derives the selection principles that were found empirically.

In the second chapter, the author describes the method used by Schrödinger, in his second paper, to obtain the equation that goes by his name. This method is based on the Hamiltonian dynamics. When it is remembered that the starting-point of Hamilton's theory was the attempt to connect the undulatory and the corpuscular theories of light, it will be evident that this furnishes a natural foundation for the new wave-mechanics. Briefly, the essential feature of Schrödinger's theory is the interpretation of Hamilton's "action," when divided by Planck's constant, h, as the phase of a wave-function. Division by h is of course an arbitrary hypothesis, but a plausible one, since its dimensions are the same as those of "action."

The author of this small volume has carried out his purpose admirably. His insistence upon the purely formal nature of the theory is particularly to be commended; while the terminology of the older physics is used, we must be careful not to carry over too far the concepts to which we have become accustomed. One who reads this book will undoubtedly wish to pursue the subject further by going to the original papers. E. P. ADAMS

The Pythagorean Proposition. By ELISHA SCOTT LOOMIS. Cleveland, Ohio. Published by the Masters and Wardens Association of the 22nd Masonic District of the Most Worshipful Grand Lodge of Free and Accepted Masons of Ohio. 214 pages. Price $2.00.

Professor Loomis states, in the foreword of his book, *The Pythagorean Proposition*, that "the object of this work is to present to the future investigator, under one cover, simply and concisely, what is known relative to the

Pythagorean proposition, and to set forth certain established facts concerning the proofs and the geometric figures pertaining thereto."

There are discussed 58 algebraic proofs, 167 geometric proofs, four quaternionic proofs and one dynamic proof. Many of these proofs admit of variations so numerous that the author states that the number of algebraic proofs, and also the number of geometric proofs, is limitless. Technically, the author is unquestionably correct in this statement. Since, however, most of the variations are produced by the introduction of unnecessary complications into an otherwise simple proof, the number of proofs worthy of serious consideration is, for most of us, certainly finite. Nevertheless, even the most critical must admit that the number of essentially distinct interesting proofs is surprisingly large, and that the realization of this fact should be a source of inspiration for both student and teacher of geometry.

Professor Loomis has produced a valuable and interesting work. Merely as an up-to-date book of reference, it is useful a contribution to mathematical literature. But it is more than a book of reference, more than a cold-blooded compilation of facts. It reflects clearly the happy individuality and the unbounded enthusiasm of its author.

The necessity of preparing for college entrance examinations, and other exigiencies of the modern secondary school program, place serious restrictions on the enthusiastic teacher. The publication of Professor Loomis's book will be amply justified if it serves no other purpose than to encourage teachers to break away from a deadly routine of formal instruction and to reclaim more of the romantic and inspirational values which are the just heritage of geometry.

RALPH D. BEETLE

College Algebra. By ARTHUR M. HARDING and GEORGE W. MULLINS. New York, The Macmillan Co., 1928. vii+324 pages.

A natural reaction to the appearance of an addition to an already large group of text books is one of curiosity concerning the purpose of and need for such addition. The authors of this *College Algebra* make no plea for the book other than that of the differing needs of colleges, the flexibility of their product, and the incorporation of some important features presumably somewhat lacking in existing college algebras.

The content, general style, and arrangement of material are much the same as those of most good elementary books on mathematics. The chapter headings include the usual ones and others not so universal, such as partial fractions, graphical representation of numbers, interest and annuities, and scales of notation. The topics are presented in an easy conversational way, for the most part by using the scheme of statement followed by illustration; and the arrangement of material on the page is excellent. Flexibility is maintained through separa-

tion of the subject into discrete divisions from which interdependence has been eliminated as far as possible without making the book seem choppy. Short tables adequate for all the numerical computations are provided, and answers to the exercises are appended.

Exclusive of the introduction, the book contains over a thousand exercises. Of these the proportion of verbal problems to purely formal drill problems is roughly one to four. They, also, are of the usual type to which some might object on the grounds of lack of imaginative stimulation for the student.

Noteworthy features of the book include those mentioned by the authors and a few others. Straightforward graphical methods are frequently employed to advantage in locating and discussing the nature of roots of polynomials, in the study of quadratic functions, and in the solutions of simultaneous equations. As an aid to graphing quadratic functions, simple derivatives are introduced; and they are further utilized in Newton's method for approximating irrational roots of polynomials. Review material is amply provided for in the introduction. A chapter on "Scales of Notation" is unusual but of doubtful value.

Among the probable objections to the book none seem to be extraordinary. The rules for finding the characteristic of the common logarithm are the customary clumsy ones and there is the usual ambiguity of statement of problems in permutations and combinations. Teachers of algebra object to the use of the term "ways" as a substitute for both "arrangements" and "groups"; they also take exception to omitting the geometrical scheme usually understood to accompany the idea of permutations in a given connection and the point of view to be assumed in counting them. Although the treatment of permutations and combinations in this book is up to par, it is evident that the authors overlooked the opportunity of relieving teachers of some of the avoidable burdens connected with what is intrinsically difficult exposition for the classroom. Problems on the change of base in logarithms and discussion of homogeneous simultaneous equations are omitted. The definition of an infinite progression is faulty, and the statement that "r^n may be made as small as one pleases by taking n larger and larger" might well be made more rigorous, even for a beginner. Furthermore, "*the* quadratic equation which has two roots" may lead beginners to believe in the uniqueness of such an equation. The student will be amused by "an airplane 900 feet high."

Aside from these details the book possesses directness and simplicity of statement. The development of the quadratic formula without the use of fractions is impressively neat; exercises concerning the roots point out several interesting relations. The idea of logarithm is clearly presented; synthetic division is clearly explained and extensively used. The outstanding feature, perhaps, is the direct use of graphs in the theory of equations and in the discussion of the quadratic function.

The conclusion arrived at by the reviewer is that although the book is not a material improvement over current algebra books and although some features of it are not wholly commendable, teachers will find it a good basis for a beginning course in the subject.

EARL L. MICKELSON

Plane and Solid Analytic Geometry. By JAMES MCGIFFERT. Ginn and Company. xiv+335 pages. Price $2.48.

This text is the result of many years of experience in teaching by the author in the Rensselaer Polytechnic Institute. It is meant as a first course in the subject and is especially suitable for freshmen.

Polar coördinates are put off until the latter part of the course, after the student has taken up in rectangular coordinates the parabola, ellipse, and hyperbola. This is not in accord with the practice in many of our best texts, where polar coordinates are taken up early and carried along concurrently with the rectangular coordinates.

Most teachers will agree with the author in not emphasizing the general equation of the second degree in the usual first course. The plan of arranging the text so that teachers may easily omit the articles on tangents and normals is also to be commended. In giving answers to all the problems, the author is doing something to help the busy teacher. Most of us do not want to take the time to determine answers when students are in doubt.

The introduction of transcendental functions in the early part of the text will give the student something of a jolt; but it has the advantage of giving the opportunity for continued application throughout the course. Difficult topics relegated to the last chapter of a text are seldom well understood.

In the opinion of the reviewer insufficient attention is given to the graphing of illustrative solved problems. This is apt to give the student the impression that graphs are not of great importance in the solution of problems. Also in discussing theory, the graph should in general be kept continually before the student's eye. The instructor should see to it that proper diagrams are always made.

The introduction of the general equation of the second degree in the chapter on the circle is to be commended. The treatment here merely shows how the general equation includes the circle as a special case. In this chapter the introduction of the derivative before the discussion of the tangent to a circle is a step in the right direction. However, the instructor must exercise patience in thus introducing some of the elements of differential calculus at this point.

This text differs from the average in having a chapter of eighteen pages in the latter part of the plane geometry part of the book entirely devoted to oblique

coördinates. Many teachers will not care to lay much stress upon this division of the subject. Freshmen usually find this type of discipline rather irksome.

The general emphasis upon theory throughout the text should be looked upon as one of the strong features of the book. In many texts there is too much cheap pedagogy with emphasis largely upon numerical illustrations.

M. O. TRIPP

Calculus of Variations. By A. R. FORSYTH. Cambridge, England, University Press, 1927. xxii+656 pages. Price $16.00.

After a ten-page "Introduction," in which the historical development of the calculus of variations from its beginnings through the time of Weierstrass is admirably sketched, the author proceeds to a consideration of integrals involving derivatives of the first order. In Chapter I are discussed integrals of the form $\int f(x, y, y')dx$, $y' = dy/dx$. The Euler, Legendre, and Jacobi tests for maxima and minima are derived by applying special weak variations, viz., those that affect only the dependent variable. Here and elsewhere throughout the book extremals are called "characteristic curves" or merely "characteristics." The word *extremal* was nowhere discovered. The theory of the first chapter is well illustrated by examples, including the Bernoulli brachistochrone and the minimum surface of revolution, which are interestingly discussed in some detail. To be noted is the case in the minimum surface problem in which the third variation has to be considered to show that an extremum does not exist. In one problem it is shown that although a minimum is provided for weak variations of the type considered, it is possible to find variations which will give the integral a still smaller value; thus the necessity of a later consideration of strong variations is shown.

Chapter II takes up the parametric form introduced by Weierstrass as applied to integrals of first order (those in which the integrand contains no derivatives of higher order than the first). Forsyth here uses the notation x_1, y_1 for dx/dt, dy/dt, respectively, and prefers to regard x as the independent, y as the dependent, variable, rather than to regard them both as dependent upon the parameter t. The minimum surface of revolution is again used for illustrative purposes. Geodesics are also employed. In this chapter the so-called corner conditions are derived.

In the third chapter, the substance of which has previously been given in a memoir by the author in the Proceedings of the Royal Society of Edinburgh, vol. 46, part 2 (1926), pp. 149–193, integrals involving second derivatives are treated. Both the simple form and the parametric form are considered. Some propositions are stated for integrals involving derivatives of order up to and including n. Terminal conditions for the case of variable end-points are fully

treated. An interesting discussion of Euler's evolute problem illustrates the general theory.

Chapters IV, V, VI are devoted to integrals of the forms $\int f(x, y, y', z, z')dx$, $\int F(x, x_1, y, y_1, z, z_1)dt$, $\int f(x, y, y', y'', z, z', z'')dx$, $\int F(x, x_1, x_2, y, y_1, y_2, z, z_1, z_2)dt$, in which accents denote differentiation with respect to x, and subscripts indicate differentiation with respect to t, of an order equal to the subscript. Results are stated, without proof, for integrals involving $n-1$ independent variables and their first derivatives with respect to the independent variable x. As illustrative examples, the problem of joining two points not in one plane with the x-axis by a curve whose first moment about that axis is a minimum, and a similar problem for moment of inertia are used.

Strong variations are introduced in Chapter VII. The Weierstrass E-function is expressed in a variety of forms, and is developed for integrals in which second order derivatives occur, for skew curves, and, in later chapters, for the isoperimetric problem and double and triple integrals. Of note is the author's proof that neither a maximum nor a minimum can exist when the parametric integrand is a rational function of the derivatives. Application is made to the problem, originally propounded by Newton, of the solid of revolution of minimum resistance (for various laws of resistance), as well as to the least action problem, the brachistochrone, geodesics, etc.

Chapter VIII is largely concerned with isoperimetric problems, including the case involving second derivatives, even in the space problem. The classic problems of finding a curve of given length which shall enclose a maximum area, and of finding a curve which when rotated will give a maximum volume with a fixed area are among those used for illustration. For exemplifying the space theory the author employs the problem of finding a curve of given length which joins two points not in one plane with a given straight line, and which has a maximum or a minimum moment about that line. The question of relative extrema restricted by equations of non-integral type is very briefly considered. An interesting illustration of this theory is his solution of the problem of finding the shortest curve of given constant circular curvature joining two points in space. He also treats the problem of geodesics on a surface $S(x, y, z)=0$ by minimizing the length integral subject to the relation $S=0$.

The next three chapters deal with double integrals. In Chapter IX a theorem on minimal surfaces, due to Schwarz, is extended so as to obtain an analytic expression of the Jacobi test regarding the conjugate of the initial boundary curve. Chapter X shows how to construct the E-function for double integrals and proves that the Weierstrass test is satisfied by minimal surfaces. It also considers the simplest case of isoperimetric problems for double integrals. These two chapters center upon the problem of minimal surfaces. The eleventh chapter treats of double integrals involving partial derivatives of second order; but,

as the author states in the preface, "there is no attempt at a full discussion, mainly because, after the application of even the simpler tests, the analysis becomes unwieldy and the developments demand the differential geometry of the curvature of surfaces."

The final chapter, number XII, deals with triple integrals. Its chief point of interest is perhaps the proof, by application of the Weierstrass E-function test, that Dirichlet's principle is not valid.

In the "Conclusion," the author mentions four questions, the investigation of which he deems desirable. The first of these has to do with conjugate points for integrals of the second order. The second is the influence of the parametric multipliers in the isoperimetric problem upon the so-called normal form of the second variation. It is often stated that an extremal surface for a double integral of first order is determined by the condition that it pass through two curves in space. The Cauchy primitive of the corresponding characteristic (Euler) equation is determinate under the condition of being tangent to a given developable along an assigned curve. Professor Forsyth's third question is whether these two determinations are equivalent and if so whether there is any limitation on the form or on the position of the final curve. Finally, he urges a study of strong variations for single integrals of second order, and of general weak variations and also strong variations for double integrals of second order.

The book seems comparatively free from typographical and other errors. The following, however, were noted:

On page 12, line 7, "increase" should be "decrease."

On page 12, line 8, "increase" should be "decrease."

On page 12, line 12, "decrease" should be "increase."

On page 12, line 13, "decrease" should be "increase."

On page 41, *Ex.* 1, y_1 should be y'. Also it is not clear that the two integrals are "effectively the same" analytically.

On page 125, the last equation should have the factor n in the last term.

On page 326, in line 5 from bottom, the letter "l" has been left out of "let."

On page 342, in line 3 from bottom, the letter "f" has been left out of "function."

On page 343, in line 2 from bottom, the letter "l" has been left out of "value."

On page 455, in the last displayed equation, the subscript r should be n.

The Weierstrass E-function is first derived by considering two neighboring characteristic curves (extremals) AP and AQ and a *straight line* PQ. It becomes necessary later to make an extension to the case in which PQ is a curvilinear arc. Circular and parabolic arcs are considered as special cases. The advantage of this method is not clear to the reviewer. It would seem more desirable to introduce the idea of a "field" of extremals, which Forsyth does not do, and to employ Hilbert's invariant integral, which he barely mentioned in an early

chapter. This same criticism applies to his development of the E-function for other problems, e.g., double integrals. The reviewer would take issue with the statement regarding the arc PQ mentioned above, that "the investigations of Weierstrass, so far as they have been made known, were limited to the case when the arc is a straight line." Both Kneser and Hancock seem to indicate that Weierstrass used a curvilinear arc in a field of extremals.

In treating double integrals some use could well have been made of papers later than those of Kobb (Acta Mathematica, vol. 16, pp. 65–140; vol. 17, pp. 321–343), notably a paper by Radon (Monatshefte für Mathematik und Physik, vol. 22, pp. 63–63). But the name of Radon does not occur in the book, nor do the names of Bliss, Carathéodory, and others who have played important rôles in advancing the theory of the calculus of variations.

Much space is devoted to ingenious but tedious transformations of the second variation in the proofs of the Jacobi condition and its extensions. This could have been avoided by employing a method developed by Bliss and described in the papers referred to in the next paragraph.

The book seems unduly long for the topics it treats, for, after all, it covers little but the elements of the subject, with the possible exception of the problem involved in integrals containing second derivatives. Practically as complete a story of the calculus of variations can be obtained by reading Bliss's delightful little book which recently appeared as number 1 of the Carus Monographs. It is the opinion of the reviewer that a book so voluminous as the one under review should be more encyclopedic in character and that it should contain something of the recent advances in the subject. Professor Forsyth states that sources which were of particular use to him are the books of Moigno-Lindelöf and Hancock. He would have done well to make more use of such books as those of Kneser, Bolza, and Hadamard. Also, two articles on recent developments in the subject (G. A. Bliss, Bulletin of the American Mathematical Society, vol. 26, pp. 343–361; and Arnold Dresden, *ibid.*, vol. 32, pp. 475–521) should have been consulted. Reference is made to the articles on the calculus of variations in the German *Encyklopädie*, but the later article in the French *Encyclopédie* is not mentioned. Practically no recent developments are included in the book, and the paucity of references would make it difficult for the inquirer to discover what is at present taking place in the domain of the calculus of variations. Several problems for research are suggested in the "Conclusion," as stated above, but aside from this the volume could hardly be called inspiring. On the other hand, it is lucid and interesting; and the author is to be commended on his treatment of many topics. In particular should be mentioned his treatment of integrals involving second derivatives, his discussion of terminal conditions, the clear manner in which he emphasizes the importance of distinguishing between extrema under weak and under strong variations, the summaries at

the ends of the chapters, and above all the excellent way in which he illustrates the theory by applications to special problems, a number of which have been specifically referred to in this review.

PAUL R. RIDER

An Introduction to the Theory of Infinite Series: By T. J. I'A BROMWICH. Second Edition, revised with the assistance of T. M. MACROBERT. London, The Macmillan Co., 1926. xv+535 pages.

The first edition of this book published in 1908, is familiar to readers of the Monthly. This second edition is largely a reproduction of the first with additional theorems and examples. Chapters I–VII have had only slight alterations. To chapter VIII there has been added a discussion of the solution of linear differential equations of the second order. Chapter X of the first edition, on "Complex series and products," becomes Chapter X on "The general theory of complex series and products" and Chapter XI on "Special series and functions."

Chapter XI of the first edition, on "Non-convergent and asymptotic series" becomes Chapter XII. The entire discussion of the theory of summable series, apart from the historical introduction, has been omitted. The discussion of asymptotic series has been enlarged. Among the additions are the asymptotic expansions of Bessel's functions, trigonometric series, Stokes's transformation, and Gibbs's phenomena.

Only slight changes have been made in the Appendices. Appendix I remains practically unchanged; appendix II now includes an account of Napier's invention of logarithms.

For those unacquainted with the first edition we add that this volume contains a mass of material some of which is not otherwise easily available to English readers. Free use is made of the calculus. Proofs are not always given. The truth of some theorems is frankly allowed to rest on the reader's intuition. Valuable as a book of reference, this volume does not meet the requirements of a class room text.

The style in the revised edition, as well as in the first edition, at no point approaches the simplicity and clarity characteristic of many of the best French mathematical works.

W. V. LOVITT

Investigations on the Theory of the Brownian Movement. By ALBERT EINSTEIN. Edited with notes by R. FÜRTH: translated by A. D. COWPER. New York, E. P. Dutton and Co., 1927. viii+124 pages.

The practice of collecting, in book form, the original papers which have played a conspicuous part in the development of an important field of scientific research is to be highly commended. Such papers are of very great value in many ways. While it is, of course, best to read them in the original language,

nevertheless a translation makes them readily available to a larger circle and hence fills a real need. Einstein's five papers on the "Brownian Movement" included in the present volume are of unusual interest and significance; they were contributed to the Annalen der Physik and the Zeitschrift für Elektrochemie during 1905–1908, and opened up an extensive field of investigation, the cultivation of which has since led to results of wide application and fundamental importance in many branches of mathematical physics and physical chemistry.

The "Brownian Movement" is the continual, irregular, zigzag motion shown by microscopic particles suspended in liquids or gases. First observed in 1827 by the English botanist Robert Brown, it remained unexplained for fifty years, when finally several investigators suggested that it might be due to the continual bombardment of the particles by the molecules of the fluid in which they were suspended. A precise test of this hypothesis first became possible as a result of Einstein's brilliant theoretical work; in the first paper here reprinted, he deduced that "bodies of microscopically-visible size suspended in a liquid will perform movements of such magnitude that they can be easily observed in a microscope, on account of the molecular motions of heat," and suggested that "it is possible that the movements . . . . discussed here are identical with the so-called 'Brownian molecular motion.' " In this and the following papers, Einstein established the now famous equations which the phenomenon should obey; but he remarked, "I will not attempt here a comparison of the slender experimental material at my disposal with the results of the theory, but will leave this comparison to those who may be handling the experimental side." The classical experiments that soon were carried out by Perrin and others completely confirmed Einstein's theoretical deductions, and incidentally provided a most direct and convincing experimental proof of the atomic hypothesis.

The final paper of the volume, "The Elementary Theory of the Brownian Movement," might well be read first in order to get a general idea of Einstein's method of procedure. His deductions are mainly thermodynamical, and it has been said that the most universal generalization that can be made about mathematical physics is that everyone finds thermodynamics a very difficult subject; furthermore, Einstein makes extensive use of the concept of "osmotic pressure," a term rather loosely employed to denote three different things. Einstein apparently uses it in the sense of diffusion pressure. His principal equations, however, have since been obtained in many alternative ways, which will be found in current treatises on physics and physical chemistry.

It is of especial interest and significance that the "Brownian Movement" introduces us to a realm in which classical thermodynamics, particularly the "Second Law," is not obeyed.

The volume opens with a brief biographical sketch; and closes with an ex-

tensive collection of helpful notes, which amplify and criticise Einstein's text and give references to much of the relevant literature.

EDGAR W. WOOLARD

Collected Papers of Srinivasa Ramanujan. Edited by G. H. HARDY, P. V. SESHU AIYAR, and B. M. WILSON. Cambridge, University Press, 1927. xxxvi+355 pages.

The papers are preceded by two sketches of the life and work of Srinivasa Ramanujan (1887–1920), the first by P. V. Seshu Aiyar and R. Ramachandra Rao and the second by G. H. Hardy. Ramanujan's career was unlike that of any modern mathematician. Until he arrived in England in 1914 he had had little access to modern works on mathematics. To quote Hardy: "Here was a man who could work out modular equations, and theorems of complex multiplication, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world, who had found for himself the functional equation of the zeta-function, and the dominant terms of many of the most famous problems in the analytic theory of numbers; and he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was. His ideas as to what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intitution, and induction, of which he was entirely unable to give any coherent account." It is therefore not surprising that a large part of his earlier work had been discovered by others. That he should have discovered them at all is in itself quite remarkable, for, as Hardy says in connection with one of Landau's theorems which Ramanujan discovered independently: "Ramanujan had none of Landau's weapons at his command; he had never seen a French or German book; his knowledge even of English was insufficient to enable him to qualify for a degree. It is sufficiently marvelous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve, and of which the solution is incomplete to the present day."

There are thirty-seven papers in all, of which seven are in collaboration with G. H. Hardy. Most of the papers deal with the theory of numbers. Ramanujan had a most amazing faculty for conjecturing theorems from empirical data. For example, from a table giving the number of unrestricted partitions $p(n)$ of n, he conjectured that if $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1$, mod δ, then

$$p(\lambda),\ p(\lambda+\delta),\ p(\lambda+2\delta),\ \cdots \equiv 0,\ \text{mod}\ \delta;$$

and he proved the theorem for a few particular cases.

The reader will be grateful to the editors for the numerous notes on Ramanujan's papers in Appendix I. Errors are pointed out and references are given to writers who commented on or made additions to Ramanujan's work.

LOUIS WEISNER

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3326. Proposed by J. J. Ginsburg, Student, Cooper Union.

Prove that, if R, r, $2s$ are the circumradius, inradius, and perimeter of a triangle, its altitudes satisfy the cubic

$$2Rx^3 - (s^2 + r^2 + 4Rr)x^2 + 4rs^2x - 4r^2s^2 = 0.$$

3327. Proposed by J. H. Neelley, Carnegie Institute of Technology.

Bisect a given line segment by means of compasses only.

3328. Proposed by Paul Wernicke, Washington, D. C.

Let A, B, C, D be four points in a plane no three of which are collinear. If AC, BD, produced, meet in F, and AD, BC meet in G, prove that $(AF/FC)/(AG/GD) = (BG/GC)/(BF/FD)$.

3329. Proposed by Lester S. Hill, Hunter College of the City of New York.

It is desired to make the following construction: N objects to be arranged in N groups, all of the same size, in such a manner that (1) every two groups have one and only one object in common, (2) corresponding to every two objects there is one and only one group which contains both. It is required to determine whether the following statement is correct or false: A necessary and sufficient condition for the existence of the required construction is that $N = 1+q+q^2$, where $q = p^r$, p denoting a prime positive integer, and r denoting a positive integer or zero.

3330. Proposed by J. B. Reynolds, Lehigh University.

A uniform elastic thread of natural length a, weight W and coefficient of elasticity e lies within and is attached to one end of a smooth tube. The tube is rigidly attached at this end to a thin vertical rod with which it makes an angle θ. Find the length of the elastic thread in steady motion under a constant angular velocity n about the vertical rod and locate the center of gravity of the thread. θ is measured clockwise from the tube to the vertical.

3331. Proposed by Otto Dunkel, Washington University.

In a spherical triangle ABC such that $AC < BC$ two points A', B', are taken on the side AB so that $\angle A'CA = \angle BCB' \leqq C/2$. Prove that $AA' \lesseqgtr B'B$ according as $AC + CB \lesseqgtr 180°$.

SOLUTIONS

263 [1917, 177]. Proposed by J. L. Riley.

To find values, positive integral, which verify the equation $x^3+2=y^2$.

Partial Solution by Lawrence Hampton, University of Oklahoma

By making use of the known facts that x^3 will be congruent to zero, one, or eight, modulo nine, and that y^2 will be congruent to zero, one, or four, modulo eight, we find that x must have the form $24n-1$. The equation is not satisfied by any positive numbers less than one hundred.

Note by Otto Dunkel, Washington University

By an extension of this process and the use of a table of squares and cubes it appears that no solution exists for a positive value of x less than 1500. It would be desirable to have a proof of the existence, or non existence, of a solution in positive integers.

274 [1917, 467]. Proposed by J. L. Riley.

Solve in positive integers the equation

$$x^4+x^3+x^2+x+1=y^2.$$

Solution by A. A. Bennett, Brown University

The complete solution of this equation in integers is given in the solution [1926, 281] of problem 2784 [1919, 366].

Also solved by Lawrence Hampton

436 [1917, 388]. Proposed by Artemas Martin, LL. D., Washington, D. C.

A circle of radius a is drawn at random on a circular slate of radius r. If another circle be drawn on the slate, what is the probability that the second circle will intersect the first?

Solution by R. E. Gaines, University of Richmond, Va.

Let O be the center of the slate with radius r, let C_4 denote the concentric circle with radius $r-a$, and C_5 the concentric circle with radius $r-3a$. Also let C_1 and C_2 denote the two circles of radius a, and C_3 the circle of radius $2a$, concentric with C_1. It will be assumed that the problem means that C_1 and C_2 shall lie wholly on the slate, and hence that the centers of C_1 and C_2 must lie within or on the boundary of C_4. First suppose that the center of C_1 falls within C_5. Then C_3 lies wholly within C_4, and if the center of C_2 falls anywhere within C_3, the circles C_1 and C_2 will intersect. The chance that the center of C_1 will fall within C_5 is $(r-3a)^2/(r-a)^2$, and the probability that the center of C_2 will fall within C_3 is $4a^2/(r-a)^2$. The probability of intersection is therefore $4a^2(r-3a)^2/(r-a)^4$.

Next suppose that the center of C_1 falls outside of C_5 but within C_4; then the center of C_2 cannot fall in that portion of C_3 which projects over C_4 but only in the area common to C_3 and C_4. Let A be the center of C_1 and C_3, and let B be a point of intersection of C_3 and C_4. Set $OA=x$, $\angle OAB=\theta$ and $\angle AOB=\phi$. The other two sides of the triangle OAB are $AB=2a$, $OB=r-a$. Then, if we take a narrow circular path of width dx, with the center at O and radius x, the chance that A will fall in this path is $2\pi x dx \div \pi(r-a)^2$; and for any particular position of A the chance that the center of C_2 will fall within C_3 is $K\div\pi(r-a)^2$, where K is the area common to C_3 and C_4. Hence the probability of intersection in this case is $2Kxdx\div\pi(r-a)^4$, where $K=(r-a)^2\phi+4a^2\theta-2S$ and $S=$ the area of OAB. To get the sum of the probabilities for all the circular paths for values of x from $r-3a$ to $r-a$, we have the following integral:

$$\frac{1}{\pi(r-a)^4}\int_{x=r-3a}^{x=r-a}[(r-a)^2\phi+4a^2\theta-2S]2xdx,$$

where $\phi=\cos^{-1}\dfrac{x^2+(r-a)^2-4a^2}{2x(r-a)}$, $\theta=\cos^{-1}\dfrac{x^2+4a^2-(r-a)^2}{4ax}$, and

S, the area of $OAB=\frac{1}{4}[\{x^2-(r-3a)^2\}\{(r+a)^2-x^2\}]^{1/2}$.

The first term of the integral gives

$$(r-a)^2x^2\phi - 2(r-a)^2S - 4a^2(r-a)^2\sin^{-1}\frac{4a^2+(r-a)^2-x^2}{4a(r-a)}.$$

The second term may be obtained from this result by replacing ϕ by θ and interchanging $r-a$ and $2a$.

The third term is

$$[4a^2+(r-a)^2-x^2]S + 4a^2(r-a)^2\sin^{-1}\frac{4a^2+(r-a)^2-x^2}{4a(r-a)}.$$

Combining these results and applying the required limits, after which certain terms may be further simplified and combined, we have as the probability of intersection in the case where A falls outside of C_6 but within C_4,

$$\frac{1}{\pi(r-a)^4}\left[2(r-a)^4\sin^{-1}\frac{a}{r-a} + 8a^2(r-a)^2\cos^{-1}\frac{a}{r-a} - 2a(r^2-2ra+3a^2)(r^2-2ra)^{1/2} - 4a^2(r-3a)^2\pi\right].$$

To this expression must be added the result previously obtained for the probability of intersection when A falls within C_5, namely $4a^2(r-3a)^2/(r-a)^4$. This will simply cancel the last term and give as the final result,

$$\frac{1}{\pi(r-a)^4}\left[2(r-a)^4\sin^{-1}\frac{a}{r-a} + 8a^2(r-a)^2\cos^{-1}\frac{a}{r-a} - 2a(r^2-2ra+3a^2)(r^2-2ra)^{1/2}\right].$$

It will be noticed that if $r=2a$ this result reduces to unity, as it should, since under this condition the circles C_1 and C_2 always intersect. If $r=3a$ the probability is $1-3^{3/2}\cdot 4^{-1}\cdot\pi^{-1}=.587$.

510 [1917, 124, 231]. Proposed by Joseph E. Rowe.

Show how to find the equation of a line perpendicular to a side of the triangle of reference and passing through a given point, in a system of homogeneous coördinates, using the condition that two lines be parallel but not the condition that two lines be perpendicular. Illustrate the method by using it to find the trilinear coördinates of the points of contact of an escribed circle of the triangle.

Solution by the Editors

Let the homogeneous coördinates (x, y, z) of a point be proportional to the distances of the point from the sides of the reference triangle ABC, and let $(1, 1,1)$ be the center of the inscribed circle. The orthocenter has the coördinates $(\sec A, \sec B, \sec C)$, and the line at infinity is $L\equiv x\sin A+y\sin B+z\sin C=0$. A line perpendicular to $z=0$ has the equation

$$\lambda(x\cos A - y\cos B) + \mu L = 0, \tag{1}$$

and it cuts the side $z=0$ in a point determined by setting $z=0$, in (1). It will also pass through the center of an escribed circle $(-1, 1,1)$, if we set $\mu=\cos A+\cos B$, $\lambda=\sin B+\sin C-\sin A$. We thus find the point of contact in this case to be $(\cos B-1)$, $(\cos A+1)$, 0. The other points of contact may be found in a similar manner.

2764 [2740; 1919, 171]. Proposed by E. H. Clarke, Hiram College.

If the coefficients of $(a-b)^k$, where k is a positive integer, be multiplied term by term by the nth powers (n being zero or a positive integer) of the terms of any arithmetic progression with common difference $d\neq 0$, the sum of the products will vanish if $n<k$; will be $(-d)^k(k!)$ if $n=k$; and, if $n=k+1$, will be the product of this last result and the sum of the terms of the arithmetic progression.

Solution by O. J. Ramler, The Catholic University of America.

The results will be obtained by expanding $(e^{ax}-e^{bx})^k$ in two ways. Expanding first the expression inside the parenthesis to terms of the second degree in x and then raising this result to the kth power, we obtain

$$(a-b)^kx^k + \tfrac{1}{2}k(a-b)^k(a+b)x^{k+1} + \cdots.$$

Now carrying out the expansion in the reverse order, we shall find for the coefficient of x^n, after setting $ka=u$ and $b-a=d$,

$$\frac{1}{n!}\sum_{i=0}^{i=k}(-1)^i{}_kC_i(u+id)^n, \qquad {}_kC_i=\frac{k!}{i!(k-i)!}.$$

Hence

$$\begin{aligned}\sum_{i=0}^{i=k}(-1)^i{}_kC_i(u+id)^n &= 0, && n<k,\\ &= k!(-d)^k, && n=k,\\ &=(k+1)!(-d)^k\,\frac{2u+kd}{2}, && n=k+1.\end{aligned}$$

2807 [1920, 80]. Proposed by S. A. Corey, Des Moines, Iowa.

Establish the identity

$$\begin{vmatrix} -x & bcy & -acu & -abv\\ y & x & -av & au\\ u & bv & x & by\\ -v & cu & cy & -x\end{vmatrix}^2 + ac\begin{vmatrix} x & bcy & -x & -abv\\ -y & x & y & au\\ u & bv & u & by\\ -v & cu & -v & -x\end{vmatrix}^2$$

$$+ ab\begin{vmatrix} x & bcy & -acu & -x\\ -y & x & -av & y\\ u & bv & x & u\\ -v & cu & cy & -v\end{vmatrix}^2 = \begin{vmatrix} x & bcy & -acu & -abv\\ -y & x & -av & au\\ u & bv & x & by\\ -v & cu & cy & -x\end{vmatrix}^2$$

Solution by the Proposer

Lagrange's like-producing quadrinomial is

$$(x^2+bcy^2+acu^2+abv^2)(p^2+bcq^2+acr^2+abs^2)=(P^2+bcQ^2+acR^2+abS^2)$$

where

$$\begin{aligned}P&=xp-bcyq-acur-abvs, & Q&=-yp-xq-avr+aus,\\ R&=up-bvq+xr+bys & S&=-vp-cuq+cyr-xs.\end{aligned}$$

Now assume for p, q, r, and s such values as to render $P=-x$, $Q=y$, $R=u$, and $S=-v$. Substitute these values of p, q, r, and s in the like-producing quadrinomial, divide by the first factor, clear of fractions, and obtain the given identity, since $q\equiv 0$.

3119 [1925, 95]. Proposed by N. A. Court, University of Oklahoma.

Through a given point to draw a line meeting the sides BC, CA, AB of a given triangle ABC in the points P, Q, R such that $PQ=kQR$, where k is a given constant.

Solution by Otto Dunkel, Washington University

Let O be the given point and OT, a variable straight line cutting the sides CB and CA in P and Q, respectively. Let the point T be taken on this variable line so that $PQ=kQT$. If we draw the ray CT and then the ray $C\infty$ parallel to OT, the cross ratio of the pencil $C(A, B, T, \infty)$ will be constant. Hence the pencils $C(T)$ and $C(\infty)$ are projective; but the latter is also projective with the pencil $O(T)$. Therefore the pencils $O(T)$ and $C(T)$ cut the side AB in two projective ranges of points. The self-corresponding points of the two ranges give the desired points R on AB. The points R may be determined by aid of an auxilliary circle as explained in 3240 Note, 1928, 46 from three pairs of corresponding points. If OT is parallel to CB, T is at infinity on CB, and this gives one pair of corresponding points on AB. A second pair may be constructed similarly by making OT parallel to CA. The third pair is constructed by taking any other position for OT. There may be two solutions, one solution, or no solution.

It is apparent from the above discussion that the locus of T is an hyperbola through O and C with asymptotes parallel to CA and CB.

3268[1927, 335]. Proposed by Tobias Dantzig, University of Maryland.

A and B are two fixed points on a conic and M and N, two variable points on the same conic, such that the cross ratio $(A, B; M, N)$ with regard to the conic remains constant. Find the envelope of the line MN.

Solution by Theodore Bennett, University of Illinois.

By making a suitable choice of trilinear coordinates the points A and B may be taken as (0, 0, 1) and (1, 0, 0), while the conic has the equation $x_2^2 - x_1x_3 = 0$.

This conic may be written parametrically $x_1 : x_2 : x_3 = t^2 : t : 1$. The points A and B are given by $t = 0$ and $t = \infty$.

Let M and N be given by the parametric values t and s; then the equation of MN is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ t^2 & t & 1 \\ s^2 & s & 1 \end{vmatrix} = 0, \text{ or } x_1 - (t+s)x_2 + stx_3 = 0,$$

But $(A, B; M, N) = (0, \infty; t, s) = t/s = k$. Substituting $t = ks$, the equation of MN becomes

$$x_1 - s(1+k)x_2 + s^2kx_3 = 0.$$

The envelope of this system of lines has the equation

$$(1+k)^2x_2^2 - 4kx_1x_3 = 0.$$

The envelope is therefore a conic which touches the given conic at A and B.

Note by the Editors.

Let $[A, B, M, N]$ denote the given cross ratio, where A, B, M, N are points on the conic (C). Consider any other point M' on (C) and let $M'N$ cut AB in K. Draw MK cutting (C) again in N'. Then the pencils $M'(A, B, M, N)$ and $M(A, B, M', N')$ are projective, and hence $[A, B, M', N'] = [A, B, M, N]$. Thus if A, B, M, N remain fixed and K varies on AB, the problem requires the envelope of $M'N'$. This is problem 3156[3152; 1925, 481] a solution of which is given in [1926, 387].

Also solved by N. A. Court and Paul Wernicke.

3270[1927, 335]. Proposed by N. A. Court, University of Oklahoma.

If a triangle is self-polar with respect to a circle, the center of the circle is the orthocenter of the triangle.

State this proposition in projective form, and give a direct proof of the proposition so stated.

Solution by Theodore Bennett, University of Illinois.

There are several ways in which the proposition may be stated, all of which amount to the same thing. The following is a fairly concise way of putting it.

Given a point C, a conic K, and a triangle $P_1P_2P_3$ which is self-polar as to K. The lines which join C to the points in which the polar line of C cuts the sides of the triangle, and the lines which join C to the vertices of the triangle form three pairs in an involution whose double rays are the tangents to K from C.

Proof: Let P_2P_3 cut the polar line of C at Q_1; then the polar line of Q_1 passes through both C and P_1, whence CP_1 and CQ_1 are harmonic to the tangents to K drawn from C. Similarly, for the other two pairs, which proves the proposition.

Also solved by Paul Wernicke.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

The twelfth annual summer meeting of the Mathematical Association of America will be held at Amherst College, Amherst, Massachusetts, on September 3 and 4, 1928. Professor B. H. Camp, of Wesleyan University, chairman of the program committee, announces that the following papers will be read at the meeting:

1. "What we mean by postulates," by Professor ALBERT A. BENNETT, Brown University.
2. "Convergence andcontinuity," by Professor ARNOLD DRESDEN, Swarthemore College.
3. "Higher forms of the icosahedron," illustrated by models, by Mr. A. HARRY WHEELER, Lecturer, Wellesley College.
4. "The past decade in cosmology," by Professor WILLIAM D. MACMILLAN, University of Chicago.
5. "Some geometrical aspects of physics," by Professor LUTHER P. EISENHART, Princeton University.
6. "The development of the Russian mathematical school," by Professor JACOB D. TAMARKIN, Brown University.

(Note: All the papers are mainly expository in character.)

A complete announcement of the meeting will be sent to members of the Association by the secretary in due time.

At their last meeting in Boston on February 24 and 25, 1928, The National Council of Teachers of Mathematics decided to publish a register of all members of the Council as a supplement to one of the forthcoming issues of "The Mathematics Teacher." This register will serve as a sort of "Whos Who?" in secondary mathematics and all such teachers of mathematics will wish to be included. The register will be sent free to all who are members at the time of publication. Any teachers of mathematics who are not now members should send in the annual dues of $2 at once to "The Mathematics Teacher," 525 West 120th Street, New York City. The payment of these dues entitles each member to a year's subscription to "The Mathematics Teacher." The address and position of each subscriber should be given.

The National Council also has published three yearbooks. The first one on "A General Survey of Progress in the Last Twenty-five Years" may be secured for $1.10 from C. M. Austin, Oak Park High School, Oak Park, Ill. The second on "Curriculum Problems in Teaching Mathematics" may be secured for $1.25 from the Bureau of Publications, Teachers College, 525 West 120th Street, New

York City. The third yearbook on "Selected Topics in Teaching Mathematics," which is just off the press, may also be obtained for $1.75 (bound volume) from the Bureau of Publications, Teachers College.

The University of Arkansas has conferred the degree of LL.D. on Professor H. E. BUCHANAN of Tulane University.

The Messel medal of the British Society of Chemical Industry has been awarded to Dr. R. A. MILLIKAN, of the California Institute of Technology, in recognition of his achievement in measuring the electrostatic charge of the electron.

On the occasion of its semi-centennial celebration, the University of Colorado conferred a doctorate of laws on Professor R. A. MILLIKAN, of the California Institute of Technology.

Dr. GEORGE SAUTÉ, of Harvard University, has been awarded one of the fellowships of the Commission for Relief in Belgium Educational Foundation, for the study of mathematics in Belgium.

It is announced that Guggenheim fellowships in astronomy, mathematics, and physics have been awarded to the following persons: to Professor PERRY BYERLY, of the University of California, for the study of mathematical geophysics; to Professor OLIVE C. HAZLETT, of the University of Illinois, for the study of the arithmetics of linear associative algebras together with their application and interpretation in other lines of mathematics, especially the theory of numbers; to Professor J. J. HOPFIELD, of the University of California, to study the Zeemann effect of the infra-red spectrum of oxygen and nitrogen; to Dr. R. J. KENNEDY, of the California Institute of Technology, for research in the theory of radiation; to Professor N. C. LITTLE, of Bowdoin College, for the study of thermo-magnetic properties of gaseous molecules; to Professor F. W. LOOMIS, of New York University, for the study of quantum mechanics; to Professor L. E. REUKEMA, of the University of California, for the theoretical and experimental study of electric discharge in gases at high frequencies; to Professor OTTO STRUVE, of the University of Chicago, for a theoretical study of the distribution and physical properties of diffuse matter in interstellar space; to Professor W. W. WATSON, of the University of Chicago, for the study of molecular spectra.

Professor PAUL ALEXANDROFF of the University of Moscow, who has been on the staff of Princeton University during the present academic year, has recently made a lecture tour of the Middle West. His tour included the Ohio State University, the University of Michigan, and the University of Iowa. At

the Ohio State University, Professor Alexandroff lectured on "Some phases of analysis situs."

The papers read at the bicentenary Newton celebration held in New York under the auspices of the History of Science Society have been published in book form with the title, *Sir Isaac Newton, 1727-1927, A Bicentenary Evaluation of his Work*. The book was prepared for publication by F. E. Brasch, and has an introduction by D. E. Smith; it is published by Williams and Wilkins, of Baltimore.

The Louisiana Academy of Sciences held its first annual meeting at Louisiana College, Pineville, May 5, 1928. Professor I. MAIZLISH, of Centenary College, is president, and Professor H. L. SMITH, of Louisiana State University, is vice-president. Professor H. A. WILSON, of Rice Institute, delivered the principal address, entitled *Recent theories of atomic structure*. Mathematical papers were read by Professors J. A. HARDIN, C. D. SMITH, H. L. SMITH, and Mrs. YETTA V. MAIZLISH. The next annual meeting will be held at the Southwestern Louisiana Institute, Lafayette.

An article by Professor C. A. Garabedian of the University of Cincinnati was recently published in the Journal de l'Ecole Polytechnique (2ᵉS., Cahier 26, 1927). The title is: *Correction de certains résultats sur la flexion d'une plaque circulaire épaisse donnér par de Saint-Venant dans la célèbre "Note finale du paragraphe 45" de la traduction de Clebsch.*

The following graduate courses in mathematics are announced for the academic year 1928–29:

Harvard University—By Professor W. F. OSGOOD: Advanecd calculus, II; Theory of functions, II. By Professor J. L. COOLIDGE: Higher geometry. By Professor E. V. HUNTINGTON: Fundamental concepts of mathematics. By Professor O. D. KELLOGG: Theory of potential functions. By Professor G. D. BIRKHOFF: Partial differential equations of mathematical physics; Advanced dynamics and the quantum theory. By Professor MARSTON MORSE: Theory of functions, I; Calculus of variations. By Professor J. L. WALSH: Introduction to modern geometry; Dynamics, II. By Professor H. W. BRINKMANN: Advanced calculus, I; Theory of groups; Differential geometry, II. By Professor M. H. STONE: Functions of real variables, II. By Dr. M. M. SLOTNICK: Differential geometry, I. Professor Walsh will conduct a fortnightly seminar on the theory of functions. Courses of research are offered by Professor Osgood in the theory of functions, by Professor Coolidge in geometry, by Professor Huntington in postulate-theory, by Professor Kellogg in the theory of potential functions, by Professor Birkhoff in the theory of differential equations, by Professor Morse in analysis situs, by Professor Walsh in analysis, by Professor Brinkmann in the theory of groups, and by Professor Stone in expansion problems.

University of Pennsylvania—By Professor E. S. CRAWLEY: Higher algebraic geometry. By Professor G. H. HALLETT: Infinite series and products, first term. Functions of a complex variable. By Professor H. B. EVANS: Quaternions and vector methods, second term. By Professor F. H. SAFFORD: Partial differential equations. By Professor H. H. MITCHELL: Analytic theory of numbers, first term. Algebraic numbers, second term. By Professor M. J. BABB: History of mathematics. By Professor F. W. BEAL: Differential equations, first term. Differential geometry, first term. By Professor J. R. KLINE: Functions of a real variable. By Professor J. D. ESHLEMAN: Synthetic projective geometry. By Professor P. A. CARIS: Higher algebra. By Professor J. M. THOMAS: Pfaff's problem, first term. Continuous groups, second term.

Professor J. D. TAMARKIN, of Brown University, has been elected a member of the American Academy of Arts and Sciences.

Professor HERMAN WEYL, of the Zurich Technical School, has been appointed to the Thomas D. Jones research professorship of mathematical physics at Princeton University.

Associate Professor J. W. ALEXANDER, of Princeton University, has been promoted to a full professorship of mathematics.

Dr. R. G. ARCHIBALD, of Columbia University, has been promoted to an assistant professorship of mathematics.

Professor DANIEL BUCHANAN, head of the department of mathematics in the University of British Columbia, has been appointed dean of the faculty of arts and sciences at that University.

Assistant Professor WILLIAM B. CAMPBELL, of Colgate University, has been appointed by the Baptist Mission Board as lecturer in mathematics at Judson College, Rangoon, Burma.

Assistant Professor CHARLES DUNCAN GREGORY, of William and Mary College, has been promoted to an associate professorship of mathematics.

Dr. LULU HOFMANN has been promoted to a lecturership in mathematics in Columbia University.

Dr. B. O. KOOPMAN, of Columbia University, has been promoted to a full professorship of mathematics.

Associate Professor S. LEFSCHETZ, of Princeton University, has been promoted to a full professorship of mathematics.

At Yale University, Professor W. R. LONGLEY has been appointed to the Colgate professorship, as successor to Professor Luquiens.

Associate Professor C. R. MACINNES, of Princeton University, has been promoted to a full professorship of mathematics.

Dr. MAX MASON, president of the University of Chicago and formerly professor of mathematical physics at the University of Wisconsin, has resigned to accept an appointment as director of the newly created division of natural sciences of the Rockefeller Foundation.

Dr. L. T. MOORE has been promoted to an assistant professorship of mathematics at Yale University.

Associate Professor G. W. MULLINS, of Barnard College, Columbia University, has been promoted to a full professorship of mathematics.

Assistant Professor OYSTEIN ORE, of Yale University, has been promoted to an associate professorship of mathematics.

Dr. D. J. STRUIK has been appointed assistant professor of mathematics at the Massachusetts Institute of Technology.

Assistant Professor J. D. TAMARKIN, of Brown University, has been promoted to a full professorship of mathematics.

Professor JOHN HASBROUCK VAN VLECK, of the department of physics of the University of Minnesota, has been appointed professor of physics at the University of Wisconsin. His work will be in theoretical physics. He is a son of Professor E. B. VAN VLECK of the department of mathematics of the University of Wisconsin.

Associate Professor J. H. M. WEDDERBURN, of Princeton University, has been promoted to a full professorship of mathematics.

The following appointments to instructorships in mathematics are announced:

Columbia University, Dr. M. S. DEMOS;

Harvard University, Dr. M. M. SLOTNICK (National Research Fellow), A. C. BERRY, A. B. BROWN, A. H. FOX, G. A. HEDLUND, J. J. HINRICHSEN, J. K. PETERSON, G. B. PRICE, G. SAUTÉ, and CHARLES WEXLER.

Dr. J. B. CHITTENDEN, professor of mathematics at the Brooklyn Polytechnic Institute, died March 20, 1928, at the age of sixty-four.

Professor GAETANO LANZA, emeritus professor of theoretical and applied mechanics at the Massachusetts Institute of Technology, died March 21, 1928, at the age of seventy-nine.

Dr. F. S. LUTHER, emeritus president and formerly professor of mathematics at Trinity College, Hartford, died January 4, 1928, at the age of seventy-seven.

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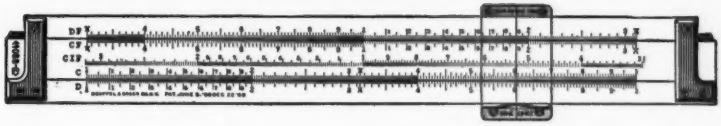

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DIRECTORY

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BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, May 4-5.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 31.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Midland College, April 28.

OHIO, Columbus, Ohio, April 5.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Dunham, N. C., March.

SOUTHERN CALIFORNIA, Fullerton Junior College, November 3.

TEXAS, Texas A. and M. College, Jan. 28.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LA.-MISS. BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE TWELFTH ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twelfth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the Colorado School of Mines, Golden, Colorado, on April 20–21, 1928. There were fifty-five present including the following twenty-five members of the association: Ralph Beatley, A. G. Clark, E. A. Cummings, I. M. DeLong, J. R. Everett, G. W. Finley, Philip Fitch, J. C. Fitterer, G. W. Gorrell, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, A. J. Lewis, G. H. Light, E. P. Martinson, S. L. Macdonald, Margaret McGinley, J. Q. McNatt, W. K. Nelson, O. H. Rechard, W. J. Risley, Mary S. Sabin, C. H. Sisam, E. B. Stouffer, C. W. Wray.

The section voted to hold its next annual meeting at the Colorado Teachers' College, Greeley, Colorado. The following officers were elected: G. W. FINLEY, chairman; G. W. GORRELL, vice-chairman; PHILIP FITCH, secretary, G. H. LIGHT, treasurer.

At a complimentary dinner given by the School of Mines on Friday, President Coolbaugh delivered an address of welcome to which Professor G. W. Finley made the response. The section was favored Friday evening by an address, "Mathematics in Italian Universities" by Professor E. B. Stouffer of the University of Kansas. Professors DeLong and Gorrell gave brief talks on the life and work of the late Professor H. E. Russell.

The following fourteen papers were read:

1. "A problem" by Mr. J. Q. MCNATT.

2. "The intersection of two special ruled cubics" by Professor SAUL POLLOCK (by invitation).

3. (a) "A method of solving quadratics" and (b) "A model for trigonometric functions" by Mr. J. W. HAZARD (by invitation).

4. "Teaching mathematics by subjects or topics" by Professor G. W. GORRELL.

5. "Note on the convergence of an infinite series" by Professor C. A. HUTCHINSON.

6. "Teaching engineering mathematics to the poor student" by Professor J. R. EVERETT.

7. "An interpretation of the imaginary resulting from an application of the principle of continuity" by Mr. PHILIP FITCH.

8. "The use of the discriminant in solving a certain type of differential equation" by Professor G. H. LIGHT.